$\boxed{\text{25 Oct 2021}}$ Approximation Algorithms for Vertex Cover

Announcement: Wilson Yoo is creating groups for Problem Set 3, should be on CMS by end of day. (sy536@cornell.edu)

We'll be looking at NP optimization problems, i.e.,

$$F(x) = \max_{y \in \mathcal{Y}(x)} V(x,y)$$

or

$$F(x) = \min_{y \in \mathcal{Y}(x)} V(x,y)$$

Some efficiently computable quantity taking values in $\mathbb{N}$.

Example.

MAX INDEP SET: given G, what is the size of the maximum independent set?

MIN VERTEX COVER: given G, what is the smallest set of vertices containing at least one endpoint of each edge?

Note the complement of an indep set is a vertex cover and vice-versa.

So if k is the max indep set size, then $n-k$ ____ min vertex cover size.

<u>Def.</u> An α-approximation algorithm for an NP maximization problem outputs some ALG such that

true optimum

$$\text{ALG} \le \text{OPT} \le \alpha \cdot \text{ALG}.$$

For minimization this changes to

$$\text{OPT} \le \text{ALG} \le \alpha \cdot \text{OPT}.$$

Note $\alpha \ge 1$, and smaller α is "better" regardless of maximization vs minimization.

Note: approximating min vtx cover doesn't imply approximating max indep set!

Knowing max ind set lies in $[n-2s, n-s]$ may not pin it down within const. factor!

Approximating weighted vertex cover

Given $G=(V,E)$ and $w: V \to \mathbb{N}$
find vertex cover $S \subseteq V$ that
minimizes $w(S) = \sum_{v \in S} w(v)$.

Or find $\min\{ w(S) \mid S \text{ a vertex cover} \}$.

First approx alg: reduce to LP.

Let x_v be a "decision variable"
whose intended meaning is $x_v = \begin{cases} 1 & \text{if } v \in S \\ \emptyset & \text{if } v \notin S. \end{cases}$

$$\text{VC-LP:} \quad \min \sum_v w(v) \cdot x_v$$

$$\text{s.t.} \quad x_u + x_v \geq 1 \quad \forall \text{ edge } (u,v)$$

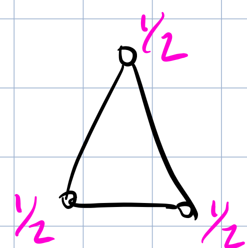
$$0 \leq x_v \leq 1 \quad \forall \text{ vertex } v.$$

The vectors in $\{0,1\}^V$ that satisfy the constraints of (VC-LP) correspond to actual VC sets and the LP obj equals $w(S)$ in that case, so

$$\min(\text{VC-LP}) \le \min\{w(S) \mid S \text{ a vtx cover}\}.$$

<u>Claim:</u> $2 \cdot \min(\text{VC-LP}) \ge \min\{w(S) \mid S \text{ a vtx cover}\}.$

Archetypical LP solution that doesn't come from a true vertex cover...



$x_v = \frac{1}{2} \quad \forall v.$

If x is any optimal vector satisfying the constraints of VC-LP, let

$$S = \{v \mid x_v \ge \tfrac{1}{2}\}.$$

Then S is a vertex cover, and

$$w(S) = \sum_{v \in S} w(v) \le 2 \cdot \sum_{v \in S} w(v)\, x_v$$

$$\le 2 \sum_{v \in V} w(v)\, x_v \quad (\text{b/c } x_v \ge 0)$$

$$= 2 \cdot \min(\text{VC-LP})$$

Approx. alg #2: primal-dual.

$$\text{(VC-LP)} \quad \begin{aligned} \min \quad & \sum_v w(v) x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad \forall e = (u,v) \\ & x_v \geq 0 \quad \forall v \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_e y_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} y_e \leq w(v) \quad \forall v \\ & y_e \geq 0 \end{aligned}$$

where $\delta(v) = \{ e \mid v \text{ is an endpoint} \}$

Idea: We know

$$\max(\text{DUAL-LP}) \leq \min(\text{VC-LP}) \leq \text{OPT}$$

So try to simultaneously find

S (vertex cover set)

y (dual LP solution)

such that $w(S) \leq 2 \cdot \left(\sum_e y_e \right)$

$$\ldots \leq 2 \max(\text{DUAL-LP}) \leq 2 \cdot \text{OPT}.$$

Algorithm:

Initialize $y_e = 0 \quad \forall e.$

Initialize $S = \emptyset$

While S is not a vertex cover:

pick some $e = (u,v)$ with neither u nor v in S.

increase y_e as much as possible

until either $\sum_{e \in \delta(u)} y_e = w(u)$ ← Put u into S

or $\sum_{e \in \delta(v)} y_e = w(v)$ ← Put v into S.

endwhile

output S

$$2 \sum_e y_e = \sum_{v \in V} \left(\sum_{e \in \delta(v)} y_e \right)$$

$$\geq \sum_{v \in S} \left(\sum_{e \in \delta(v)} y_e \right) = \sum_{v \in S} w(v) = w(S).$$