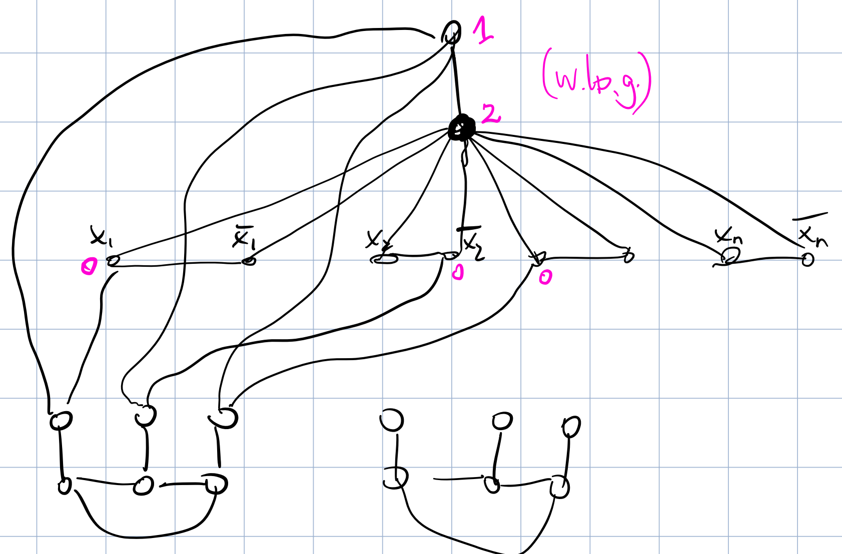
$\boxed{\text{22 Oct 2021}}$ 3-colorability and Max Cut

Def. A proper 3-coloring of $G = (V, E)$ is a function $h: V \to \{0,1,2\}$ such that $h(u) \neq h(v)$ $\forall$ edge (u,v).

3-COLORABILITY is the decision problem: given G is it 3-colorable?

3-SAT $\leq_p$ 3-COLORABILITY

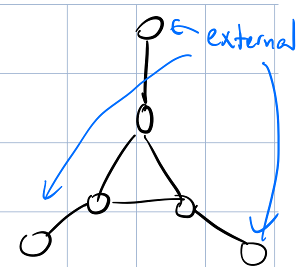
Gadget to represent choosing a truth value for each variable.



0 = false
1 = true

Representing clauses.

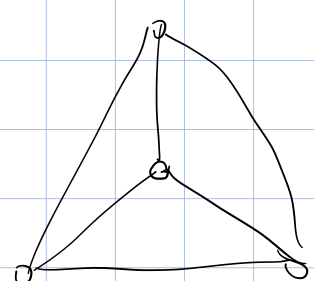
Observation.... in the graph



if the 3 external vertices are all given the same color, then there's no way to extend to a proper coloring of all 6 vertices.

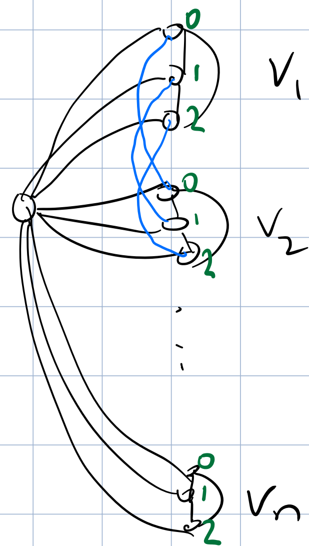
Otherwise a proper coloring is possible.

3-COLORABILITY $\leq$ MAX CUT.



This graph has exactly 3 max cuts (modulo swapping A and B)
(3 partitions into 2 sets of 2 vertices)

Graph with n vertices:



A vertex is represented by

1 anchor node (common to all vtx gadgets)

3 color nodes (specific to one vtx gadget)

6 black edges

An edge (v_1, v_2) represented by

3 blue edges

Easy to see: A cut can separate at most 4 black edges out of 6 in each vertex gadget

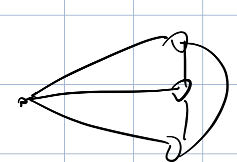
.... and if it separates 4 out of 6 in each, then it can separate ~~at~~ most 2 out of 3 in each edge gadget.

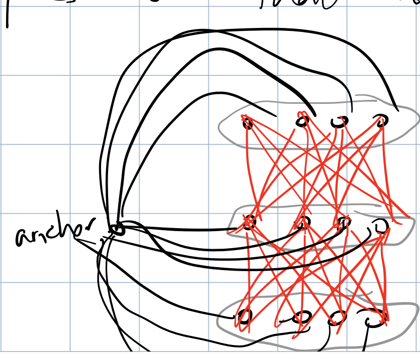
We also saw: There is a transformation (called T_{AB} in prev lecture) that takes a proper 3-coloring and converts it to a cut that splits

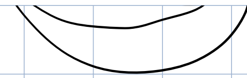
$4n$ black
$+2m$ blue

edges.

A better vertex gadget:
make each of the 3 "color nodes" into M copies of that node.
$(M > m)$.







Lemma. There are 3 ways to arrange this gadget, with anchor on A side, so that $2M + 2M^2$ edges are cut.

Any other cut with anchor on A side cuts $\leq 2M + 2M^2 - M$ edges.

Proof. Let S_1, S_2, S_3 be the 3 bundles of M nodes corresponding to the 3 colors.

Let a_1, a_2, a_3
b_1, b_2, b_3
be the # of nodes on the A side, B side for each of S_1, S_2, S_3.

E.g. $a_3 = |A \cap S_3|$.

Then # cut edges is

$$b_1 + b_2 + b_3 + a_1 b_2 + a_1 b_3$$
$$+ a_2 b_1 + a_2 b_3 + a_3 b_2 + a_3 b_1$$

and

$$a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{N}$$

and

$$a_i + b_i = M \qquad (i = 1, 2, 3).$$

Use same edge gadget as before. (3 blue edges).

Then ask if $\exists$ a cut the has $\geq (2M + 2M^2)n + 2m$ edges in it.