20 Oct 2021 — Some NP reductions

<u>Announcement</u>: No RDK office hours today.

First example: $3SAT \le_p$ STRICT-4-SAT

exactly 4 literals per clause

Reduction introduces 1 extra variable for each clause of the 3-SAT formula.

E.g. if $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_4)$

Reduction outputs

$$\phi' = \left[\begin{array}{l} (x_1 \vee x_2 \vee x_3 \vee y_1) \wedge (x_1 \vee x_2 \vee x_3 \vee \bar{y}_1) \\ \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_4 \vee y_1) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_4 \vee \bar{y}_1) \end{array} \right]$$

Properties of this reduction:

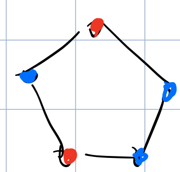
- linear time
- preserves satisfiability.

$\vec{x}$ satisfies $\phi \iff \exists \vec{y}$ s.t. (x,y) satisfies ϕ'.

$\iff \forall \vec{y}$ (x,y) satisfies ϕ'.

Next example: 3 SAT $\leq_p$ INDEP SET

Def: An independent set in a graph (V,E) is $S \subseteq V$ such that no edge has both its endpoints in S.



indep't

not indep't.

INDEP SET: given $G=(V,E)$ and $k \in \mathbb{N}$ does G have an indep set with $\geq k$ vertices?

3 parts of a reduction $A \leq_p B$.

Encoding A in the form of B

$\boxed{\text{Input of } A} \xrightarrow{R} \boxed{\text{Input of } B}$

Witness of A $\xrightarrow{T_{AB}}$ Witness of B

Decoding

Witness of A $\xleftarrow{T_{BA}}$ Witness of B

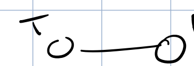
If A and B are defined by verifiers V_A and V_B then reduction R is correct if

$$(1) \quad \forall_{x,y} \quad V_A(x,y)=1 \Longrightarrow V_B\big(R(x), T_{AB}(y)\big)$$
$$(2) \quad \forall_{x,z} \quad V_B(R(x),z)=1 \Longrightarrow V_A\big(x, T_{BA}(z)\big)$$

Game plan for $3SAT \leq_p$ INDEP SET.

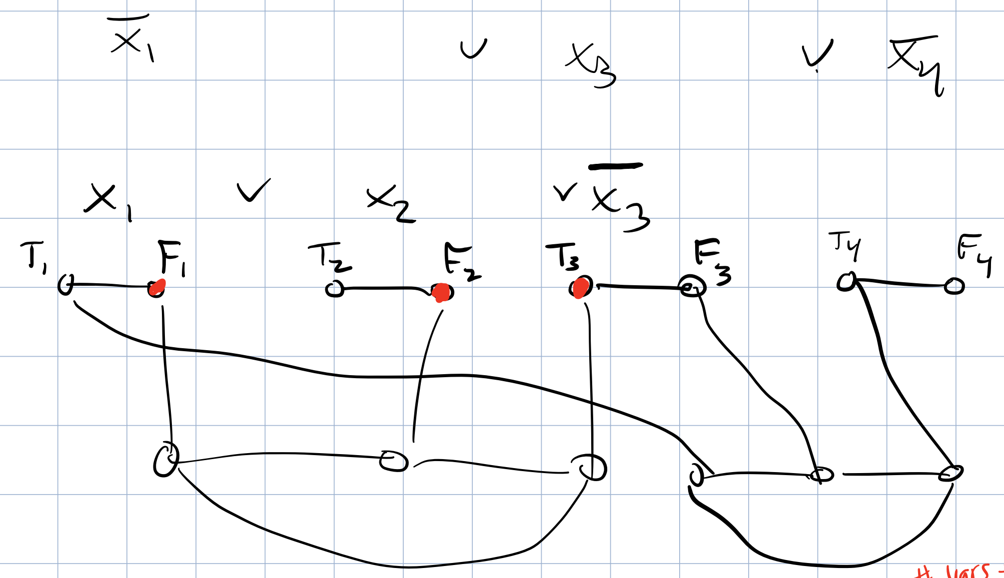
variable $\longmapsto$ subgraph with two max. indep sets; S_T, S_F.

$x_i \longmapsto$



truth assignment of a variable $\longmapsto$ corresponding subset S_T or S_F.

clause $\longmapsto$ structure that "rewards" satisfying truth assignments by letting us enlarge the indep set.



Set target indep set size, $k = \overset{\#\text{ vars }+}{n} + \overset{\#\text{ clauses}}{m}$

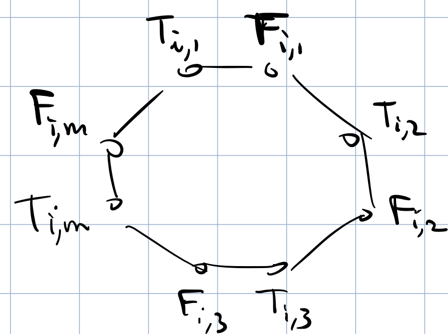
An indep set of $k = n+m$ elements must have <u>exactly</u> one vertex in each gadget.

So T_{BA} decodes a truth assignment from a k-element indep set S by setting $x_i = \begin{cases} T & \text{if } T_i \in S \\ F & \text{if } F_i \in S. \end{cases}$

IND-SET(3): The special case of indep set when the graph has maximum degree 3. (each vertex is in ≤ 3 edges).

Idea: Redesign "variable gadget" to be a cycle of length $2m$

x_i



Clause gadget remains the same.

Clause j attaches to nodes whose 2nd subscript is j.

$$k = n \cdot m + m$$

Coming up Friday: MAX-CUT.
Given undirected $G = (V, E)$ and $k \in \mathbb{N}$
is there a partition $V = A \cup B$
such that at least k edges
have one endpoint in A, the other in B.