15 Oct 2021

Ellipsoid Algorithm

Recap of LP Duality:

$$\max\left\{ c^T x \,\middle|\, Ax \le b \right\}, = \min\left\{ b^T y \,\middle|\, A^T y = c,\ y \ge 0 \right\}$$

Suppose we want to solve

$$\max\left\{ c^T x \,\middle|\, Ax \le b,\ x \ge 0 \right\}.$$

Then letting $\mathbb{1}$ denote $n \times n$ identity, $x \ge 0$ can be written $-\mathbb{1} x \le 0$.

$$\boxed{\max\left\{ c^T x \,\middle|\, Ax \le b,\ x \ge 0 \right\}} = \max\left\{ c^T x \,\middle|\, \begin{bmatrix} A \\ -\mathbb{1} \end{bmatrix} x \le \begin{bmatrix} b \\ 0 \end{bmatrix} \right\}$$

$$= \min\left\{ \begin{bmatrix} b & 0 \end{bmatrix}^T \begin{bmatrix} y \\ z \end{bmatrix} \,\middle|\, \begin{bmatrix} A^T & -\mathbb{1} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = c,\ \begin{bmatrix} y \\ z \end{bmatrix} \ge 0 \right\}$$

$$= \min\left\{ b^T y \,\middle|\, A^T y - z = c,\ y \ge 0,\ z \ge 0 \right\}$$

$$= \boxed{\min\left\{ b^T y \,\middle|\, A^T y \ge c,\ y \ge 0 \right\}}$$

Next unfinished business...

LP feasibility $\leq_p$ LP search $\leq_p$ LP optimization

Now to show: LP optimization $\leq_p$ LP search

To solve $\max\{c^Tx \mid Ax \leq b\}$
search for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$
satisfying

$$\mathcal{P}' = \{Ax \leq b,\ A^Ty = c,\ y \geq 0,\ c^Tx = b^Ty\}.$$

In proving LP duality we showed that
if $\{x \mid Ax \leq b\} \neq \emptyset$ and $\max\{c^Tx \mid Ax \leq b\} < \infty$
then $\mathcal{P}' \neq \emptyset$ and any $(x,y) \in \mathcal{P}'$ satisfies

$$c^Tx = \max\{c^Tx' \mid Ax' \leq b\}$$

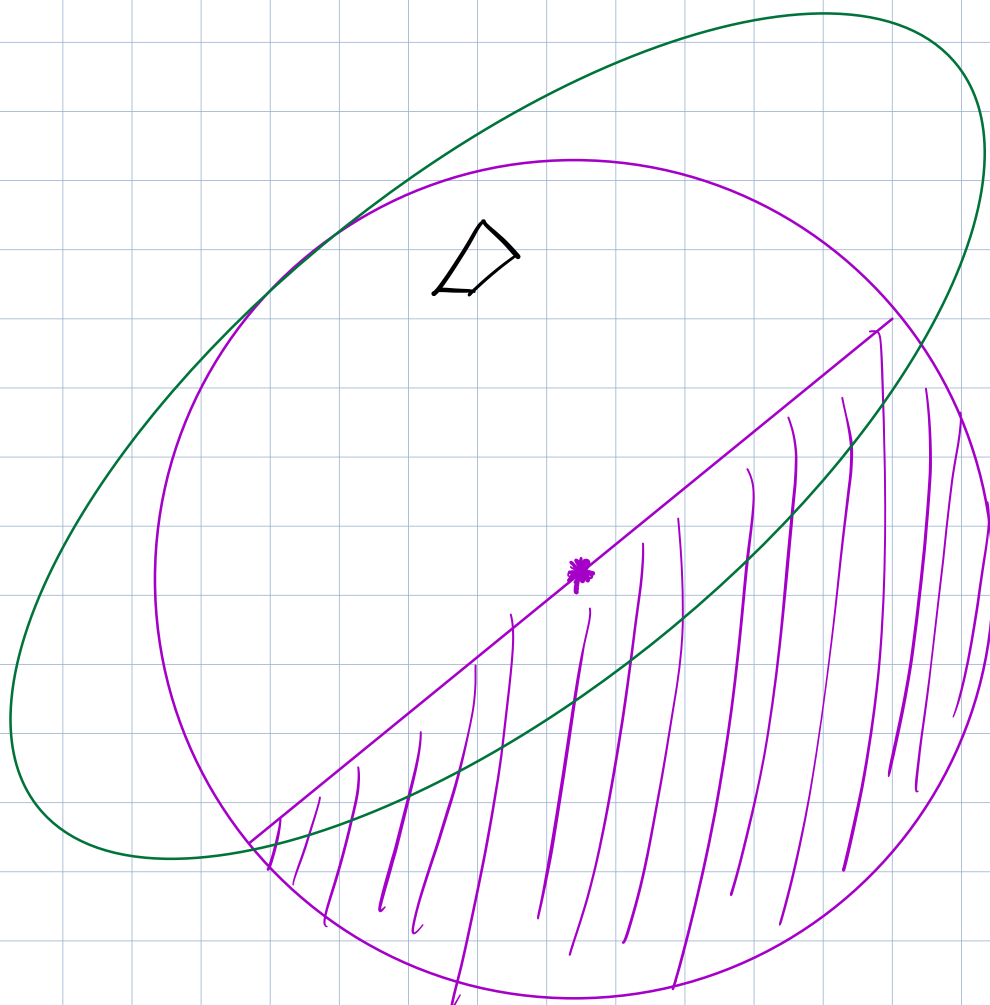
In other words searching for $(x,y) \in \mathcal{P}'$
has two outcomes:

- find $(x,y) \in \mathcal{P}' \Rightarrow x$ is optimal for $\max\{c^Tx \mid Ax \leq b\}$.
- discover $\mathcal{P}' = \emptyset \Rightarrow$ maximum in LP opt. is undefined.

Solving LP search in poly time.

Idea: "Multidimensional binary search."

Trying to find $x \in P$.



Def. An ellipsoid in $\mathbb{R}^n$ is the image of the unit ball under an invertible affine transformation. Equivalently it is a set that can be described by

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \;\middle|\; \| T(x - x_0) \|_2 \leq 1 \right\}$$

for some invertible $n \times n$ matrix T and point $x_0 \in \mathbb{R}^n$ called the center of $\mathcal{E}$.

Def. A <u>separation oracle</u> for a convex set $P \subseteq \mathbb{R}^n$ is a subroutine that accepts a query point x and returns either:

(i) assertion $x \in P$.

(ii) pair $(a, b) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$P \subseteq \mathcal{H}(a,b) = \{ x \mid a \cdot x \leq b \}$$

but $x \notin \mathcal{H}(a,b)$.

Suppose we are given a "numerical complexity bound", β.
(Think: β is # of digits in the input data)

Then ellipsoid alg runs in poly(n,β) time assuming:

(1) Separation oracle queries can be answered in poly(n,β) time

(2a) We are given ellipsoid $\mathcal{E}_0 \supseteq P$ such that $\mathrm{vol}(\mathcal{E}_0) \leq 2^{\mathrm{poly}(n,\beta)} \cdot \mathrm{vol}(P)$.

<u>**OR**</u>

(2b) We are promised that the separation oracle always outputs $(a,b) \in \mathbb{Z}^n \times \mathbb{Z}$

$$\|a\|_\infty, |b| < 2^{\mathrm{poly}(n,\beta)}$$

Lemma. If $\mathbb{B}^n = \{ x \in \mathbb{R}^n \mid \|x\|_2 \leq 1 \}$
and
$$\mathcal{H} = \{ x \in \mathbb{R}^n \mid x_1 \geq 0 \}$$

then $\mathbb{B}^n \cap \mathcal{H} \subseteq \mathcal{E}$ where

$$\mathcal{E} = \{ x \in \mathbb{R}^n \mid \| D \cdot (x - \tfrac{1}{n+1} \hat{e}_1) \| \leq 1 \}$$

and
$$D = \left(\frac{n+1}{n}\right)^2 \begin{pmatrix} 1 & & & & 0 \\ & 1-\frac{2}{n+1} & & & \\ & & 1-\frac{2}{n+1} & & \\ & & & \ddots & \\ 0 & & & & 1-\frac{2}{n+1} \end{pmatrix}$$

and $\mathrm{vol}(\mathcal{E}) < \exp\left(-\frac{1}{n+1}\right) \cdot \mathrm{vol}(\mathbb{B}^n)$.

Ellipsoid $\mathcal{E}'$
Halfspace $\mathcal{H}'$ thru $\mathrm{ctr}(\mathcal{E}')$

transf T $\quad (\mathcal{E}', \mathcal{H}') \mapsto (\mathbb{B}^n, \mathcal{H})$

$\mathcal{E}'' \supseteq \mathcal{E}' \cap \mathcal{H}' \xleftarrow{T^{-1}} \mathcal{E}$