13 Oct 2021   LP Duality

Recap: $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$

Simplex solves

$$\max \{c \cdot x \mid x \in P\}.$$

by walking from vertex to vertex along edges of P until we reach a local optimum.

Actually consider $\mathbb{R}[\varepsilon]$ as an ordered vector space over $\mathbb{R}$.

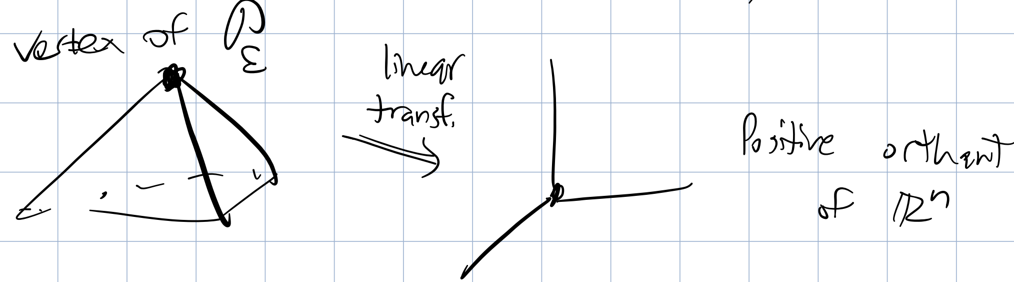
Let $P_\varepsilon = \{x \in \mathbb{R}[\varepsilon]^n \mid Ax \leq b + \vec{\varepsilon}\}$

$$\vec{\varepsilon} = \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \varepsilon^3 \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

We saw last time every vertex of P_ε has exactly n tight constraints, not more.

Simplex walks on vertices of P_ε until it hits a local opt, then projects that point $x \in \mathbb{R}[\varepsilon]^n$ back to $\mathbb{R}^n$ by setting $\varepsilon = 0$, and evaluating polynomials coordinatewise.

(Notation: $x, x' \in \mathbb{R}[\varepsilon]^n$ $\quad x \approx_\varepsilon x'$ if x and x' map to same point of $\mathbb{R}^n$ when we set $\varepsilon = 0$.)



Recall if $J \subseteq [m]$ is the index set for the tight constraints at vertex x_J then

$$A_J x_J = b_J + \varepsilon_J$$

rows of matrix A indexed by J

subvector of b indexed by J

subvector of ε indexed by J.

$$x_J = A_J^{-1} b_J.$$

$$P_\varepsilon = \{x \mid Ax \le b + \varepsilon\} \subseteq \{x \mid A_J x \le b_J + \varepsilon_J\}.$$

Setting $\boxed{w = b_J + \varepsilon_J - A_J x}$

we have $w \ge 0$ for all $x \in P_\varepsilon$

$w = 0$ at x_J

and a small neighborhood of x_J maps to a small neighborhood of $\vec{0}$ in

$$Q_\varepsilon = \{ w \in \mathbb{R}[\varepsilon]^n \mid w_i \geq 0 \ \forall i \}$$

Objective function $c \cdot x$
under the change of variables
becomes an affine function of w.
(linear plus a constant)

$$x = A_J^{-1}(b_J + \varepsilon_J - w).$$

$$c^T x = c^T A_J^{-1}(b_J + \varepsilon_J - w)$$

$$= c^T A_J^{-1}(b_J + \varepsilon_J) - c^T A_J^{-1} w$$

Let $y_J = (A_J^T)^{-1} c$, so $y_J^T = c^T A_J^{-1}$.

$$c^T x = y_J^T(b_J + \varepsilon_J) - y_J^T w$$

$w = \vec{0}$ is a local maximum of the function $f(w) = y_J^T(b_J + \varepsilon_J) - y_J^T w$ in Q_ε
if and only if $\frac{\partial f}{\partial w_i} \leq 0 \ \forall i$.

$$\iff y_J \geq 0.$$

When simplex terminates we have found
a vector $y_J \in \mathbb{R}[\varepsilon]^n$ such that

$$A_J^T y_J = c \qquad y_J \geq 0 \qquad b_J^T y_J \approx_\varepsilon c^T x_J$$

Also $c^T x_J = y_J^T (b_J + \varepsilon_J) \approx_\varepsilon y_J^T b_J = b_J^T y_J$

Now define $y \in \mathbb{R}^m$ by setting

$$y_i = \begin{cases} (y_J)_i & \text{evaluated at } \varepsilon = 0 \text{ if } i \in J \\ 0 & \text{if } i \notin J, \end{cases}$$

Writing coordinates in J before those not in J,

$$y \approx_\varepsilon \begin{bmatrix} y_J \\ \hline 0 \end{bmatrix}$$

$$A^T y \approx_\varepsilon \left[A_J^T \,\middle|\, A_{[m]\setminus J}^T \right] \begin{bmatrix} y_J \\ \hline 0 \end{bmatrix} = A_J^T y_J = c$$

$$y \geq 0$$

$$b^T y \approx_\varepsilon \left[b_J^T \,\middle|\, b_{[m]\setminus J}^T \right] \begin{bmatrix} y_J \\ \hline 0 \end{bmatrix} = b_J^T y_J \approx_\varepsilon c^T x_J$$

Simplex termination condition lets us discover a vector $y \in \mathbb{R}^m$ s.t.

$$A^T y = c, \quad y \geq 0, \quad b^T y = c^T x_J.$$

Now I claim x_J is an optimal LP solution.

If $x \in P$ i.e. $Ax \leq b$ then

$$\boxed{c^T x = \underset{\uparrow}{\ } y^T A x \underset{\uparrow}{\leq} y^T b = b^T y} \underset{\uparrow}{=} c^T x_J$$

(using $A^T y = c$) (using $y \geq 0$) (using $b^T y = c^T x_J$)

Summary. LP Duality Theorem

For any non-empty polyhedron $P = \{Ax \leq b\}$ such that $\max\{c^T x \mid x \in P\}$ is finite,

$$\max\{c^T x \mid x \in P\} = \min\{b^T y \mid A^T y = c,\ y \geq 0\}.$$

This proves max ≤ min (WEAK DUALITY)

Simplex algorithm yields a pair x_J, y certifying max = min.

Simplex can take exponential time!

Klee-Minty cubes.

<u>Cube:</u>

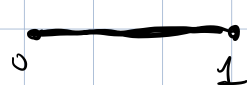
$$0 \le x_1 \le 1$$
$$0 \le x_2 \le 1$$
$$0 \le x_3 \le 1$$
$$\vdots$$
$$0 \le x_n \le 1$$

<u>KM cube:</u>

$$0 \le x_1 \le 1$$
$$\delta x_1 \le x_2 \le 1 - \delta x_1$$
$$\delta x_2 \le x_3 \le 1 - \delta x_2$$
$$\vdots$$
$$\delta x_{n-1} \le x_n \le 1 - \delta x_{n-1}$$

$\left(\delta < \frac{1}{2} \right)$

$n = 1$



$n = 2$