Recap: \[ P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \]

Simplex solves \[ \max \{ c \cdot x \mid x \in P \} \]
by walking from vertex to vertex along edges of \( P \) until we reach a local optimum.

Actually consider \( \mathbb{R}^\varepsilon \) as an ordered vector space over \( \mathbb{R} \).

Let \[ P_\varepsilon = \{ x \in \mathbb{R}^\varepsilon^n \mid Ax \leq b + \varepsilon e \} \]
where \( \varepsilon = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n \end{bmatrix} \)

We saw last time every vertex of \( P_\varepsilon \) has exactly \( n \) tight constraints, not more.

Simplex walks on vertices of \( P_\varepsilon \) until it hits a local opt, then projects that point \( x \in \mathbb{R}^\varepsilon^n \) back to \( \mathbb{R}^n \) by setting \( \varepsilon = 0 \), and evaluates coordinates coordinatewise.
(Notation: $x, x' \in \mathbb{R}^n, x \sim x'$ if $x$ and $x'$ map to same point of $\mathbb{R}^n$ when we set $\varepsilon = 0$.)

Consider if $J = \{1, \ldots, m\}$ is the index set for the tight constraints at vertex $x_J$

Then,

\[ A_J x_J = b_J + \varepsilon_J \quad \text{subvector of } \varepsilon \text{ indexed by } J. \]

\[ x_J = A_J^{-1} b_J. \]

\[ \mathcal{P}_{\varepsilon} = \left\{ x \mid A x \leq b + \varepsilon \right\} \subseteq \left\{ x \mid A_J x = A_J^{-1} b_J + \varepsilon_J \right\}. \]

Setting \[ w = b_J + \varepsilon_J - A_J x \]

we have $w \geq 0$ for all $x \in \mathcal{P}_{\varepsilon}$

\[ w = 0 \quad \text{at} \quad x_J \]

and a small neighborhood of $x_J$ maps to a small neighborhood of $0$ in
\[ Q_e = \{ x \in \mathbb{R}^n | w_i > 0 \, \forall i \} \]

**Objective Function** \( C \cdot x \)

under the change of variables becomes an affine function of \( w \).

(linear plus a constant)

\[ x = A^{-1}_j (b_j + e_j - w). \]

\[ c^T x = c^T A^{-1}_j (b_j + e_j - w) \]

\[ = c^T A^{-1}_j (b_j + e_j) - c^T A^{-1}_j w \]

Let \( y_j = (A^{-1}_j)^T c \), so \( y_j^T = c^T A^{-1}_j \),

\[ c^T x = y_j^T (b_j + e_j) - y_j^T w \]

\( w = 0 \) is a local maximum of the function \( f(w) = y_j^T (b_j + e_j) - y_j^T w \) in \( Q_e \)

if and only if \( \frac{\partial f}{\partial w} \leq 0 \, \forall i \).

\[ \iff y_j > 0. \]

When simplex terminates we have found a vector \( y_j \in \mathbb{R}^n \) such that
\[
A_j^T y_J = c
\]
\[
y_J = 0
\]
\[
b_j^T y_J \approx c x_j
\]

Also
\[
c_{x_J} = y_J^T (b_J + \varepsilon_J) \approx y_J^T b_J = b_j^T y_J
\]

Now define \( y \in \mathbb{R}^m \) by setting
\[
y_i = \begin{cases} (y_J)_i & \text{if } i \in J \\ \varepsilon & \text{if } i \notin J. \end{cases}
\]

Writing coordinates in \( J \) before those not in \( J \),
\[
y \approx \begin{bmatrix} y_J \\ \varepsilon \end{bmatrix}
\]

\[
A_j^T y \approx \begin{bmatrix} A_j^T \\ 0 \end{bmatrix} \begin{bmatrix} y_J \\ \varepsilon \end{bmatrix} = A_j^T y_J = c
\]
\[
y = 0
\]
\[
b_j^T y \approx \begin{bmatrix} b_j^T (b_J + \varepsilon_J) \\ 0 \end{bmatrix} \begin{bmatrix} y_J \\ \varepsilon \end{bmatrix} = b_j^T y_J \approx c x_j
\]
Simplex termination condition lets us discover a vector $y \in \mathbb{R}^m$ st.
\[ A^T y = c, \quad y \geq 0, \quad b^T y = c^T x_f. \]

Now I claim $x_f$ is an optimal LP solution.

If $x \in P$, i.e., $Ax \leq b$ then

\[ c^T x = y^T A x \leq y^T b = b^T y = c^T x_f \]

(using $A^T y = c$) (using $y \geq 0$) (using $b^T y = c^T x_f$)

**Summary:** LP Duality Theorem

For any non-empty polyhedron $P = \{ Ax \leq b \}$ such that $\max \{ c^T x \mid x \in \mathbb{R}^n \}$ is finite,

\[ \max \{ c^T x \mid x \in \mathbb{P} \} = \min \{ b^T y \mid A^T y = c, \ y \geq 0 \}. \]

This proves $\max = \min$ (Weak Duality)

Simplex algorithm yields a pair $x_f, y$ certifying $\max = \min$. 
Simplex can take exponential time!

Klee-Minty cubes

**Cube:** \[ 0 \leq x_i \leq 1 \]
\[ 0 \leq x_j \leq 1 \]
\[ 0 \leq x_k \leq 1 \]
\[ \vdots \]
\[ 0 \leq x_n \leq 1 \]

**RM cube:** \[ 0 \leq x_i \leq 1 - \delta x_i \]
\[ \delta x_i \leq x_j \leq 1 - \delta x_i \]
\[ \delta x_2 \leq x_3 \leq 1 - \delta x_2 \]
\[ \vdots \]
\[ \delta x_{n-1} \leq x_n \leq 1 - \delta x_{n-1} \]

\( \delta < \frac{1}{2} \)