8 Oct 2021 — Simplex Algorithm and LP Duality

To-do list.

1. WLOG $\vec{0}$ is a vertex of P.
2. Recap local search procedure.
3. Deal with "degeneracy": $|J(x)| > n$.
4. Prove correctness

   a.k.a "strong LP duality."

$$P = \{x \mid a_i \cdot x \le b_i \quad \forall i = 1, 2, \ldots, m\}$$

$$\max\{c \cdot x \mid x \in P\}.$$

<u>Simplex alg:</u> Start at any vertex x of P

while $\exists$ x' neighboring x s.t.

  $c \cdot x' > c \cdot x$

  update $x \leftarrow x'$

endwhile

output x

The WLOG assumption is no longer $\vec{0} \in P$.
It is that P is nonempty and we can efficiently find $x \in P$ vertex.

Step 1. WLOG $\{x_i \geq 0 \ \forall i\}$ are among the constraints of P.

Given any LP with n variables $z_1, \dots, z_n$ modify to LP with $2n$ variables

$x_1, \dots, x_n \geq 0$

$y_1, \dots, y_n \geq 0$

Substitute $x_i - y_i$ for z_i in each constraint of original LP. Also same substitution in objective func.

Step 2. Solve a "preliminary LP" to find an initial vertex of P.

If
$$P = \left\{ x \;\middle|\; \begin{array}{ll} a_i \cdot x \leq b_i & \forall i = 1, \dots, m \\ x_j \geq 0 & \forall j = 1, \dots, n \end{array} \right\}$$
let
$$Q = \left\{ (x, s) \;\middle|\; \begin{array}{ll} a_i \cdot x - s \leq b_i & \forall i = 1, \dots, m \\ x_j \geq 0 & \forall j = 1, \dots, n \\ s \geq 0 & \end{array} \right\}$$

Solve $\min s$ st. $(x, s) \in Q$.

$x = 0$, $s = 0 \vee \max_i \{-b_i\}$ is a vtx of Q.

If $s > 0$ at optimality, P is empty.

If $s=0$, then x is a vertex of P.

(2) Vertices, adjacency, finding an adjacent vertex where $c \cdot x' > c \cdot x$.

Recap. $\{a_i \cdot x \leq b_i\}$ can be summarized as

$$Ax \leq b \qquad A = \begin{bmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \\ - a_m^T - \end{bmatrix}$$

Coordinatewise $\leq$ relation on vectors.

For $J \subseteq [m]$ let A_J, b_J denote the submatrix (subvector) of A (b) obtained by selecting the rows indexed by J.

$$\mathcal{B}(A) := \{ J \mid A_J \text{ is an invertible square matrix} \}$$

For $J \in \mathcal{B}(A)$, the constraints $\{a_i \cdot x \leq b_i \mid i \in J\}$ are simultaneously tight at one unique vector

$$x_J = A_J^{-1} b_J$$

and this is a vertex of P if it belongs to P.

Two vertices $x_J, x_{J'}$ are quasi-adjacent if $|J \cap J'| = n-1$.

This means either $x_J, x_{J'}$ lie along an edge of P

or $x_J = x_{J'}$.

Adjacent: quasi-adjacent <u>and</u>

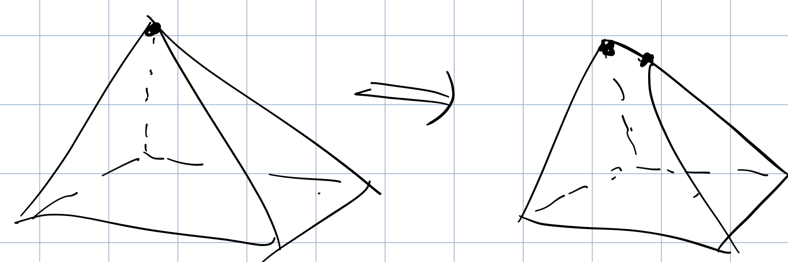
$$A_J^{-1} b_J \neq A_{J'}^{-1} b_{J'}$$

Given vertex $x = x_J$ if we want to find all quasi-adjacent vertices $x_{J'}$ we do the following

for all $i \in J$
 for all $i' \in [m] \setminus J$
  let $J' = (J \setminus \{i\}) \cup \{i'\}$
  if $\det(A_{J'}) \neq 0$
   $x_{J'} = A_{J'}^{-1} b_{J'}$
   is quasi-adj to x_J.

General case of LP reduces to "non-degerate" case where every vertex has exactly n tight constraints, not more, by "perturbation":

Take $Ax \le b$, perturb to $Ax \le b + \vec{\varepsilon}$



Make b into a vector with coordinates in $\mathbb{R}[\varepsilon]$, ordered by comparing polynomials lexicographically.

$$b + \vec{\varepsilon} = \begin{bmatrix} b_1 + \varepsilon \\ b_2 + \varepsilon^2 \\ b_3 + \varepsilon^3 \\ \vdots \\ b_m + \varepsilon^m \end{bmatrix}$$

If there is $\vec{x} \in \mathbb{R}[\varepsilon]^n$ that satisfies $A_J \vec{x} = (b+\varepsilon)_J$ then $|J| \le n$.

This is because every coordinate of the vector $A_J \vec{x}$ is a linear combination (with $\mathbb{R}$ coeffs) of $x_1, x_2, \ldots, x_n$.

The coords of $A_J \vec{x}$ lie in an n-dim'l linear subspace of $\mathbb{R}[\varepsilon]$.

But the coords of $\vec{b+\varepsilon}$ are all linearly indep.

$\Longrightarrow |J| \leq n$.

Conclusion: simplex algorithm always terminates (finds a locally optimal vertex).