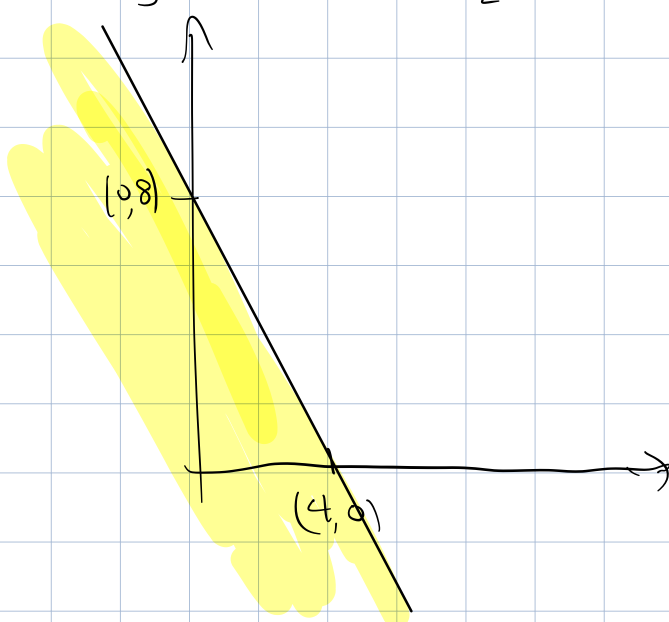
6 Oct 2021 Linear Programming and Simplex Algorithm

<u>Def.</u> A closed <u>halfspace</u> in $\mathbb{R}^n$ is the solution set of an inequality $a \cdot x \leq b$.

$$H = \{ x \in \mathbb{R}^n \mid a \cdot x \leq b \}$$

E.g. $2x_1 + x_2 \leq 8$



<u>Def.</u> A polyhedron is the intersection of a finite # of halfspaces.

<u>Def.</u> Linear programming refers to any of the following problems where one is given, as an input a polyhedron P.

(i) [LP feasibility] Is $P \neq \emptyset$?

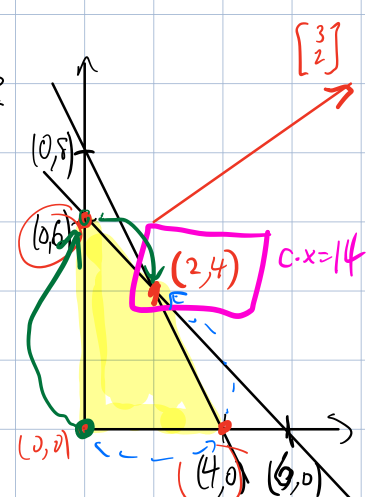
(ii) [LP search] If $P \neq \emptyset$ output $x \in P$. Else output $\perp$.

(iii) [LP optimization] Given $c \in \mathbb{R}^n$, if $P \neq \emptyset$ output $x \in P$ that maximizes $c \cdot x$ else output $\perp$.

Clearly (i) $\leq_p$ (ii) $\leq_p$ (iii).
Later we'll see (iii) $\leq_p$ (i) so the 3 problems are computationally equivalent.

(Running) example:

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$



A polyhedron in $\mathbb{R}^n$ defined by $O(n)$ inequalities (halfspaces) may have 2^n vertices.

E.g. $x_1 \geq 0 \quad x_2 \geq 0 \quad \cdots \quad x_n \geq 0$

$x_1 \leq 1 \quad x_2 \leq 1 \quad \cdots \quad x_n \leq 1$

Solution set $[0,1]^n$ has 2^n vertices.

For now assume $\vec{0}$ is a vertex of P. (Later we'll reduce the general case to this special case.)

SIMPLEX ALGORITHM. Take a walk on the vertex set of P starting from $\vec{0}$ and moving from suboptimal vertices x to adjacent vertices x' s.t. $c \cdot x' > c \cdot x$. Stop when there is no such adj. vertex. ("locally optimal") Output the current x.

How to find?

If more than one neighboring x' satisfies $c \cdot x' > c \cdot x$ the heuristic for choosing x' is called a <u>pivot rule</u>.

<u>vertex of P:</u> If P is defined by halfspace $H_i = \{ x \in \mathbb{R}^n \mid a_i \cdot x \leq b_i \}$ for $i = 1, 2, \ldots, m$, then a vertex of P is x satisfying $a_i \cdot x = b_i$ for indices $i \in J(x) \subseteq [m]$ such that $\{ a_i \mid i \in J(x) \}$ contains a basis of $\mathbb{R}^n$.

<u>adjacent vertices:</u> x and x' are adjacent if $J(x, x') = \{ i \mid a_i \cdot x = a_i \cdot x' = b_i \}$ is such that $\{ a_i \mid i \in J(x, x') \}$ spans an $(n-1)$-dimensional subspace of $\mathbb{R}^n$.

Finding x' adjacent to x such that $c \cdot x' > c \cdot x$, if such an x' exists...

Here, it's helpful if we assume $|J(x)| = n$. Simply try every set J' obtained from $J(x)$ by removing one element, inserting another.