27 Sept 2021 Network Flow: Intro

A <u>flow</u> in a directed graph with vertex set V is a function

$$f : V \times V \to \mathbb{R}$$

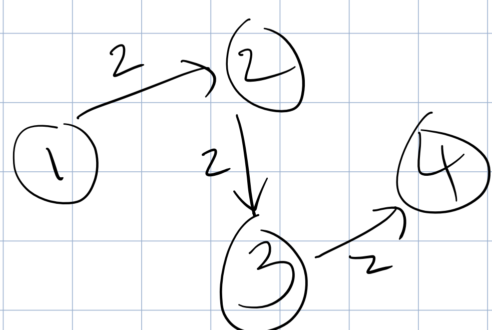
that satisfies

[skew-symmetry] $\quad f(v,u) = -f(u,v) \quad \forall u,v$

It is called an s-t flow, for specified vertices s ("source"), t ("sink") if it additionally satisfies

[s-t flow conservation]

$$\forall u \notin \{s,t\} \quad \sum_{v \in V} f(u,v) = 0$$



means $\begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$

If $G = (V, E)$ is a directed graph and $c : E \to \mathbb{R}$ is a function ("capacity") extend c to a function $V \times V \to \mathbb{R}$ by

$$c(u,v) = \begin{cases} c(u,v) & \text{if } (u,v) \in E \\ 0 & \text{if } (u,v) \notin E. \end{cases}$$

Then we say flow f is <u>feasible</u> with respect to c if it satisfies

[capacity constraints] $\quad f(u,v) \leq c(u,v) \quad \forall u,v \in V.$

<u>Def.</u> The value of flow f, denoted by $|f|$, is the sum

$$\sum_{v \in V} f(s,v).$$

("Net flow out of s.")

For a partition of V into A, B with $s \in A$, $t \in B$ we call A, B

an s-t cut and define

$$f(A,B) = \sum_{u \in A} \sum_{v \in B} f(u,v)$$

$$c(A,B) = \sum_{u \in A} \sum_{v \in B} c(u,v)$$

Lemma. If f is an st flow and (A,B) is an s-t cut then

$$f(A,B) = |f|.$$

Proof. Induction on # elements in A. When we enlarge A to $A \cup \{w\}$ we modify the sum by

- subtracting $f(u,w)$ $\forall u \in A$
- $\rightarrow$ adding $f(w,u)$ $\forall u \in A$
- adding $f(w,v)$ $\forall v \in B$.

Conservation constraint says $\sum_{u \in A} f(w,u) + \sum_{v \in B} f(w,v)$ equals zero.

Cor. If f is an st flow feasible w.r.t. capacities c, then $\forall$ st cut (A,B)

$$|f| = f(A,B) \leq c(A,B)$$

Hence the capacity of <u>every</u> s-t cut is an upper bd on the value of <u>every</u> s-t flow.

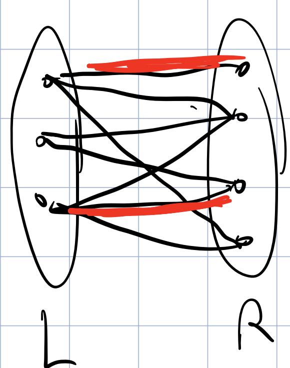
$$(\text{max flow}) \leq (\text{min cut})$$

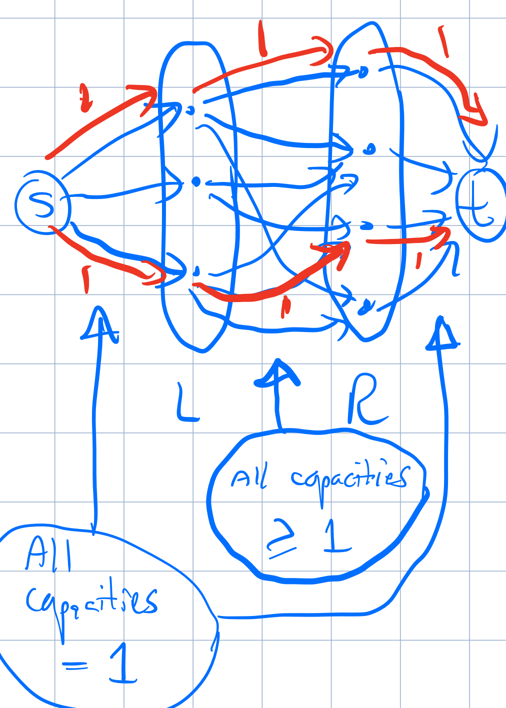
Main result in this lecture:

$$\text{max flow} = \text{min cut}$$

for every finite flow network.

Reducing bipartite matching to flow:



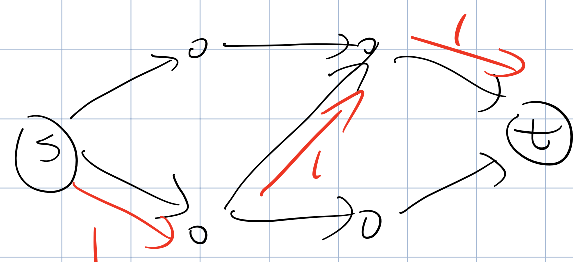


Def. Given a network $G=(V,E)$ with capacities $c: V\times V \to \mathbb{R}$ and flow $f: V\times V \to \mathbb{R}$ the residual network $G_f = (V, E_f)$ has capacities

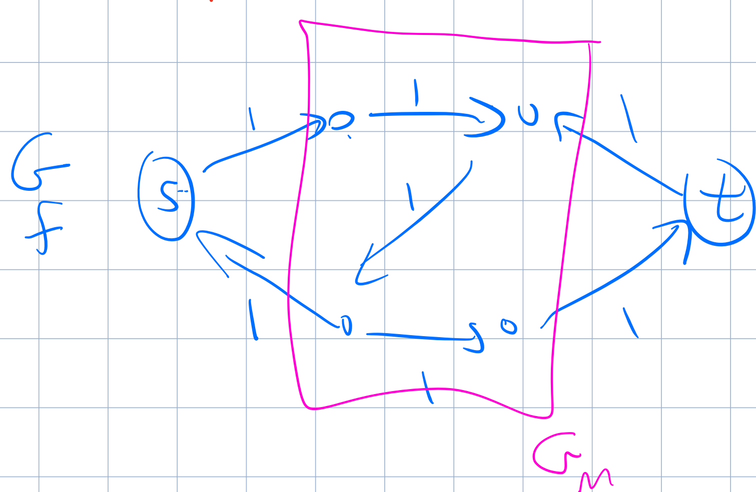
$$c_f(u,v) = c(u,v) - f(u,v)$$

and edge set $E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$.

E.g.



All capacities $= 1$.



Lemma: Let $G=(V,E)$ be a flow network with capacities $c: V\times V \to \mathbb{R}$, and let $f: V\times V \to \mathbb{R}$ be an s-t flow. There is a bijection

$$\left\{\begin{array}{l}\text{flows } f' \text{ in } G \\ \text{such that } f+f' \\ \text{is feasible w.r.t. } c\end{array}\right\} \qquad f'$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\left\{\begin{array}{l}\text{flows } f'' \text{ in } G_f \\ \text{such that } f'' \\ \text{is feasible w.r.t. } c_f\end{array}\right\} \qquad f''$$

$f+f'$ feasible w.r.t. c

$\Updownarrow$

$\forall u,v \quad f(u,v)+f'(u,v) \leq c(u,v)$

$\Updownarrow$

$\forall u,v \quad f'(u,v) \leq c(u,v) - f(u,v)$

$= c_f(u,v)$

$\Updownarrow$

f' feasible w.r.t. c_f.

Corollary. The following are equivalent

1. f is not a maximum flow
2. In G_f there is a flow f' such that $|f'| > 0$.
3. In G_f there is a path from s to t.

Proof. ~~Hardest~~ step is showing

$$\{\text{no } s\text{-}t \text{ path in } G_f\} \Rightarrow \left\{\begin{array}{c}\text{no flow of} \\ \text{pos. value}\end{array}\right\}$$

Let $A = \left\{\begin{array}{c}\text{vertices reachable in } G_f \\ \text{by a path from } s\end{array}\right\}$

$B = V \setminus A$

By assumption $s \in A$, $t \in B$.

$$\text{max flow}(G_f) \leq c(A,B) = 0.$$

Max Flow - Min Cut Theorem:

If f is a maximum flow $\exists$ s-t cut (A,B) s.t. $|f| = c(A,B)$.

Proof. We just saw that in G_f $\exists$ an st cut (A,B) with

$$c_f(A,B) = 0$$

$$c(A,B) - f(A,B) = 0$$

$$c(A,B) = f(A,B) = |f|.$$