$\boxed{17 \text{ Sep } 2021}$ Matchings and Determinants, Part II

RECAP. $G = (L, R, E)$ bipartite

$|L| = |R| = n$

$B = n \times n$ matrix of formal variables

$$B_{ij} = \begin{cases} x_{ij} & \text{if } (i,j) \in E \\ 0 & \text{if not} \end{cases}$$

$\det(B) \neq 0$ iff G has a perfect matching

Schwartz-Zippel Lemma.

if $P(x_1, \ldots, x_m)$ is a nonzero polynomial with coefficients in a field $\mathbb{F}$ and $X_1, \ldots, X_m$ are sampled indep at random from $S \subseteq \mathbb{F}$ with $|S| = s$,

$$\Pr\left(P(X_1, \ldots, X_m) = 0 \right) \leq \frac{md}{s}$$

where d is the max exponent of any variable in any monomial.

<u>Proof</u> Induction on m.

Base case: $m=0$.

P is a nonzero constant, so

$$\Pr(P=0) = 0. \quad ✓$$

Induction step: $m>0$.

$(c \le d)$

$$P = \sum_{i=0}^{c} Q_i(x_1,\dots,x_{m-1}) \cdot x_m^i$$

If $P(X_1,\dots,X_m)=0$ one ~~of~~ of two things must happen.

① $Q_c(X_1,\dots,X_{m-1})=0$

Prob $\le \frac{(m-1)d}{S}$ by induct hyp.

② $Q_c(X_1,\dots,X_{m-1}) \ne 0$ but

$P(X_1,\dots,X_{m-1},X_m)=0$.

Then X_m is one of the roots of

$$R(x) = \sum_{i=0}^{c} Q_i(X_1,\dots,X_{m-1}) \cdot x^i$$

This equation has $\le c$ solutions

So $\Pr(X_m$ is among the roots$) \le \frac{c}{S} \le \frac{d}{S}$

Exercise: What goes wrong in the proof if the polynomial P is defined over a ring rather than a field?

$$P(x_1, x_2, x_3) = x_1 x_2 x_3$$

in the ring $\mathbb{Z}/(8)\,[x_1, x_2, x_3]$.

Sample x_1, x_2, x_3 unif from $\{0, 2, 4, 6\}$

Def. Let ω be the smallest constant such that two $n \times n$ matrices can be multiplied in time $O(n^{\omega+\varepsilon})$ for all $\varepsilon > 0$.

For remainder of lecture $O^*(n^\omega)$ denotes "$O(n^{\omega+\varepsilon})$ $\forall \varepsilon > 0$."

Deciding if $\det(A) \neq 0$ in $O^*(n^\omega)$.

First obs: $\det(A) \neq 0 \iff \det(A^T A) \neq 0$.

From now on, C is a symm pos semidefinite matrix.

Attempt to invert C.

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \longleftarrow \tfrac{n}{2} \times \tfrac{n}{2} \text{ square blocks.}$$

$$\begin{pmatrix} \mathbb{1} & 0 \\ -C_{21}C_{11}^{-1} & \mathbb{1} \end{pmatrix} C = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} - C_{21}C_{11}^{-1}C_{12} \end{pmatrix}$$

$$\begin{pmatrix} \mathbb{1} & 0 \\ -C_{21}C_{11}^{-1} & \mathbb{1} \end{pmatrix} C \begin{pmatrix} \mathbb{1} & -C_{11}^{-1}C_{12} \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} - C_{21}C_{11}^{-1}C_{12} \end{pmatrix}$$

$T(n)$ = time to invert SPD matrix

$$T(n) \le 2 \cdot T\left(\tfrac{n}{2}\right) + O^*(n^\omega)$$

$$= O^*(n^\omega).$$

$$2 \le \omega < 2.373$$

Example algorithmic applications:

1. Triangle detection in $O^*(n^\omega)$ time.
2. k-clique detection in

$$O^*\left(\binom{n}{k/3}^\omega\right) \leq O^*\left(n^{(k/3)\cdot\omega}\right)$$
$$< O(n^{0.8k}).$$

Mucha & Sankowski (2006) improved on Lovasz's algorithm to be able to solve the perfect matching search problem in randomized $O^*(n^\omega)$ time.

<u>Fact:</u> $\exists$ perfect matching containing edge (i,j)
$\iff$ [the (j,i) entry of B^{-1} is nonzero and $B_{ij} \neq 0$]

<u>Proof.</u>
$$(B^{-1})_{ji} = \pm \frac{\det(B \text{ with row } i, \text{ col } j \text{ deleted})}{\det(B)}$$