$\boxed{\text{15 Sep 2021}}$ Finish Online Matching

Start Algebraic Algorithms for Matching

3 announcements

1. Prof K office hours tomorrow (Thurs) canceled.
2. TA Makis Arsenis will have Zoom office hrs every Mon. 1-2 starting next week. (See Ed + 6820 website for link.)
3. Problem 4: use Hopcroft-Karp. ($\exists$ $O(m\sqrt{n})$ also for bipartite max matching.)

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Finishing online matching.

Fractional matching: algorithm WATERFILLING.
Analysis by (complicated!) application of primal-dual method.

Randomized matching: algorithm RANKING.
Also $\frac{e}{e-1}$ competitive.
Can also be analyzed by (complicated)

application of primal-dual.
(See typeset lecture notes.)

This factor $\frac{e}{e-1}$ truly is best possible for fractional & randomized algorithms. (§4.2 of lecture notes.)

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Matchings and Determinants

<u>Recall:</u> for a square matrix $A$,

$$\det(A) = \sum_{\text{perm } \sigma} (-1)^{\operatorname{sgn}(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

So if $A$ is the "bipartite adj matrix" of a bipartite graph with $|L| = |R| = n$, i.e.

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ \emptyset & \text{if not} \end{cases}$$

then permutations $\sigma$ correspond to ways of pairing vertices on $L$ with partners on $R$.

$$\prod_{i=1}^{n} A_{i,\sigma(i)} = \begin{cases} 1 & \text{if } \sigma \text{ represents a PM} \\ \emptyset & \text{if not.} \end{cases}$$

So, the number of perfect matchings is

$$\text{per}(A) = \sum_{\text{perm } \sigma} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

Evaluating $\det(A)$ takes at most $O(n^3)$ time.

Evaluating $\text{per}(A)$ is the complete problem for complexity class #P (which is at least as hard as NP, probably strictly harder).

The fact that $\det(A) \equiv \text{per}(A) \mod 2$ means you can decide in poly time if bipartite graph has even or odd # perfect matchings.

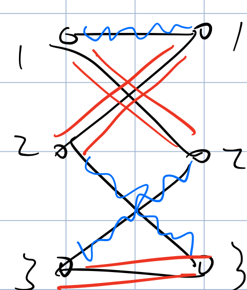
Lovasz (1979): use determinants and randomization to decide if $G$ has a perfect matching. (fast)

Let $x_{ij}$ be a formal variable corresponding to edge $(i,j) \in E$.

Define matrix $B$ whose entries are $\emptyset$'s and formal variables...

$$B_{ij} = \begin{cases} x_{ij} & \text{if } (i,j) \in E \\ \emptyset & \text{if not} \end{cases}$$

Then $\det(B)$ is a multivariate polynomial in $\{x_{ij}\}$.

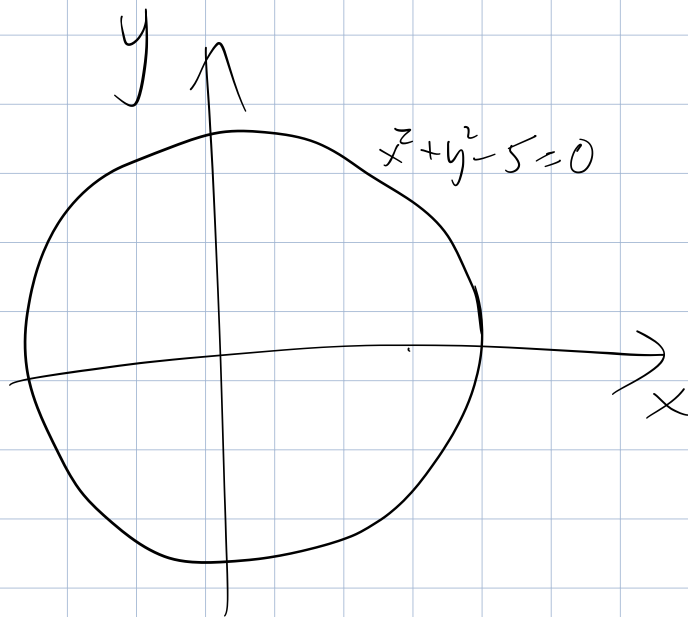
E.g. $G =$ 


$$B = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 0 & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix}$$

$$\det B = -\underline{x_{11}x_{23}x_{32}} - \underline{x_{12}x_{21}x_{33}}$$

So $\det(B)$ has exactly one monomial for each perfect matching.

$\det(B) \equiv 0$ iff $G$ has no perf matching!




## Schwartz-Zippel Lemma.

If $P(x_1, \ldots, x_m)$ is a nonzero polynomial with coefficients in a field $\mathbb{F}$ and $X_1, \ldots, X_m$ are sampled indep at random from $S \subseteq \mathbb{F}$ with $|S| = s$,

$$\Pr\left( P(X_1, \ldots, X_m) = 0 \right) \leq \frac{md}{s}$$

where $d$ is the max exponent of any variable in any monomial.