13 Sep 2021 — Online matching: fractional and randomized

Announcement

- Groups for Problem Set 1 are assigned on CMS now.
- If you try to contact a group member and they're non-responsive let us know sooner rather than later.
- On future problem sets your group will have the option to remain matched if you all want that.

ONLINE (FRACTIONAL) MATCHING

Reminder: $G = (L, R, E)$ $R = [T]$.

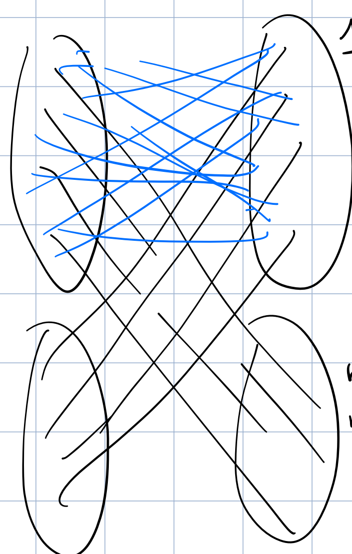
When vertex $j \in R$ "arrives" at time j, the algo learns its set of neighbors, $N(j) \subseteq L$ and needs to:

- match j with one neighbor or leave it permanently unmatched.
- (fractional) choose values $x_{ij} \geq 0$ such that $\sum_{i \in L} x_{ij} \leq 1$, $x_{ij} = 0$ for $i \notin N(j)$

$$\sum_{j \in R} x_{ij} \le 1 \quad \forall i.$$

Naive idea for fractional matching:
each j sets $x_{ij} = \dfrac{1}{|N(j)|}$ $\forall\, i \in N(j)$

if this doesn't exceed the capacity constraint of any neighbor.



1 : splits $\frac{1}{n+1}$ units to each nbr.
2 : same thing
⋮
n : same thing

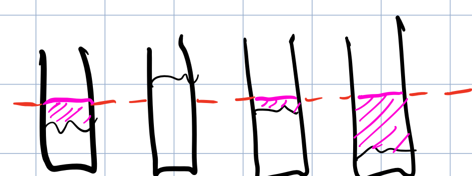
n+1 : assigns $\frac{1}{n+1}$ to unique nbr.
n+2 : – – – –

Sum up: vertices in R assign
$$n + \frac{n}{n+1} < n+1$$
units, while max matching has size $2n$.

"Waterfilling algorithm": imagine each $i \in L$ as a bucket of water. When j arrives the level of water in bucket i is

$$d_{j-1}(i) = \sum_{k=1}^{j-1} x_{ik}$$

In continuous time, a "scan line" sweeps from level ϕ to level 1 and j contributes water to buckets whose level is below scan line until 1 unit total has been allocated.



$$x_{1j} + x_{2j} + x_{3j} + x_{4j} = 1$$

Let $n_j(t) = \#\{ i \in N(j) \mid d_{j-1}(i) < t \}$.

Let $\hat{\ell}(j)$ (the "water level" of j) be the unique solution of

$$\int_0^{\hat{\ell}(j)} n_j(t)\, dt = 1.$$

$$\ell(j) = \min\{1, \hat{\ell}(j)\}.$$

$$x_{ij} = \max\{0, \ell(j) - d_{j-1}(i)\}$$

maximize $\displaystyle\sum_{(i,j)\in E} x_{ij}$

subject to $\displaystyle\sum_{j} x_{ij} \leq 1 \quad \forall i$ (scale by y_i)

$\displaystyle\sum_{i} x_{ij} \leq 1 \quad \forall j$ (scale by y_j)

$x_{ij} \geq 0 \quad \forall (i,j)\in E$

$$\sum_{j} y_i x_{ij} \leq y_i \qquad \text{if } y_i \geq 0.$$

$$\underbrace{\sum_{i} y_j x_{ij} \leq y_j}_{} \qquad \text{if } y_j \geq 0$$

$$\sum_{(i,j)\in E} (y_i + y_j) x_{ij} \leq \sum_{i\in L} y_i + \sum_{j\in R} y_j$$

$$\sum_{(i,j)\in E} x_{ij} \leq \sum_{i\in L} y_i + \sum_{j\in R} y_j \qquad \text{if } y_i + y_j \geq 1 \ \forall (i,j)\in E.$$

Plan of attack: enhance WATERFILLING to calculate y_i's and y_j's as it runs, so that in the end:

$$y_i, y_j \geq 0 \qquad \forall i, j$$

$$y_i + y_j \geq 1 \qquad \forall (i,j) \in E$$

$$\sum_{(i,j)\in E} x_{ij} \geq \frac{e-1}{e}\left[\sum_{i\in L} y_i + \sum_{j\in R} y_j\right]$$

Proposal: Let $g(z) = \frac{e^z - 1}{e-1}$.

(1) $g(0) = 0$, $g(1) = 1$

(2) g is increasing on $[0,1]$

(3) $1 - g(z) + g'(z) = \frac{e}{e-1}$ for all $z \in [0,1]$.

$$1 - g(z) = \frac{e - e^z}{e-1} \qquad g'(z) = \frac{e^z}{e-1}$$

Let $y_i = g(d(i)) = g\left(\sum_j x_{ij}\right)$

$y_j = 1 - g(\ell(j))$.

$y_i, y_j \geq 0$ ✓

IF $(i,j) \in E$ we need $y_i + y_j \geq 1$.

$$y_i + y_j = 1 - g(\ell(j)) + g(d(i)).$$

$$d(i) \geq \ell(j)$$
$$g(d(i)) \geq g(\ell(j))$$

$$1 - g(\ell(j)) + g(d(i)) \geq 1.$$

As scan line rises from t to $t+dt$

$\sum_{i,j} x_{ij}$ increases by $\eta_j(t) \cdot dt$

$\sum_{i \in L} y_i$ increases from t to $t+dt$

$\therefore$ increases by

$$\eta_j(t) \cdot g'(t)\, dt$$

$\sum_{j \in R} y_j$ increases by $\eta_j(t) \cdot [1 - g(t)]\, dt$