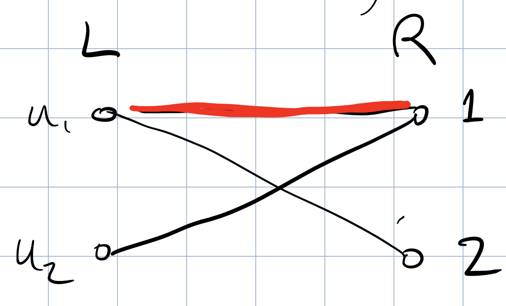
$\boxed{\text{10 Sep 2021}}$ # Online Matching

Bipartite graph $G = (L, R, E)$ whose edge set is unknown!

$R = [T] = \{1, 2, \dots, T\}$.

At time $t \le T$, vertex $t \in R$ arrives and reveals its set of neighbors $N(t) \subseteq L$. Algorithm must immediately decide

- to add an edge $(i,t)$ to $M$ for some $i \in N(t)$
- not to add any edge containing $t$. (in that case $t$ will never be matched).




You can't always pick a max matching online...

We say an algorithm is (strictly) c-competitive if its matching, $M_{alg}$, and the maximum matching, $M_{opt}$, always obey

$$\#M_{opt} \leq c\,(\#M_{alg})$$

for every input sequence. For randomized algo, this becomes

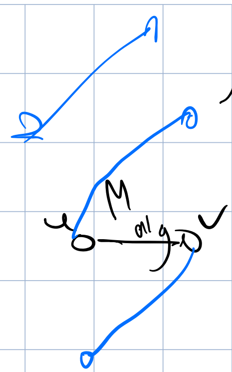
$$\#M_{opt} \leq c \cdot \mathbb{E}[\#M_{alg}].$$

Example above shows that deterministic strictly $c$-competitive algorithms do not exist for $c < 2$.

**Def.** An algorithm is greedy if it always matches vertex $t$ when possible.

**Lemma.** Any greedy alg. is strictly 2-competitive.

**Proof.** To prove, we will construct a function $h: M_{opt} \to M_{alg}$ and show $h$ is $(\leq 2)$-to-1.

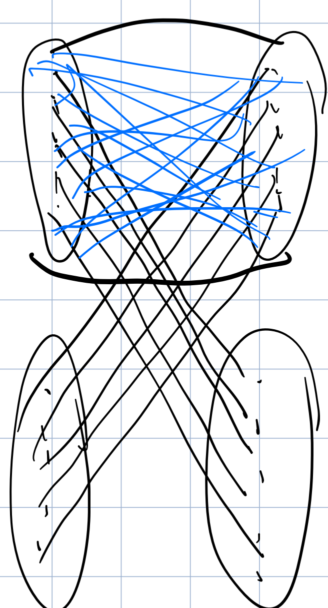



$h(e) \triangleq$ any edge in $M_{alg}$ that shares an endpoint with $e$.

If $e=(u,v)\in M_{alg}$, then $h$ maps at most 2 edges in $M_{opt}$ to $e$: the edge of $M_{opt}$ that has $u$ as endpt + 〃 〃 〃 〃 $v$ 〃.

GREEDY WITH UNIFORM RANDOMIZATION CAN DO BADLY.

n n



n n

As $n \to \infty$ $\quad \mathbb{E}[\# M_{alg}] \sim n$

$\# M_{opt} = 2n$

What to do?

RANKING works as follows.

At time $\emptyset$, choose random permutation of $L$.

At time $t > \emptyset$ when vertex $t \in R$ arrives, match to the unmatched neighbor that occurs earliest in the random permutation.