Online Matching

Bi-partite graph $G = (L, R, E)$ whose edge set is unknown!

$R = \{1, 2, \ldots, T\}$

At time $t \leq T$, vertex $t \in R$ arrives and reveals its set of neighbors $N(t) \subseteq L$. Algorithm must immediately decide:
- to add an edge $(ij, t)$ to $M$ for some $i \in N(t)$
- not to add any edge containing $t$ (in that case $t$ will never be matched).

We say an algorithm is (strictly) $c$-competitive if its matching, $M$, and the maximum matching, $M^*$, always obey:

You can't always pick a max matching online...
\[ \#M_{\text{opt}} \leq c \left( \#M_{\text{alg}} \right) \]

for every input sequence. For randomized alg., this becomes

\[ \#M_{\text{opt}} \leq c \cdot \mathbb{E}[\#M_{\text{alg}}]. \]

Example above shows that deterministic strictly c-competitive algorithms do not exist for \( c < 2 \).

**Def.** An algorithm is greedy if it always matches vertex \( t \) when possible.

**Lemma.** Any greedy alg. is strictly 2-competitive.

**Proof.** To prove, we will construct a function \( h : M_{\text{opt}} \to M_{\text{alg}} \) and show \( h \) is \((\leq 2)\)-to-1.
\( h(e) \triangleq \) any edge in \( M_{\text{alg}} \) that shares an endpoint with \( e \).

If \( e = (u, v) \in M_{\text{alg}} \), then \( h \) maps at most 2 edges in \( M_{\text{opt}} \) to \( e \): the edge of \( M_{\text{opt}} \) that has \( u \) as endpoint.

Greedy with uniform randomization can do badly.

As \( n \to \infty \), \( \mathbb{E}[\# M_{\text{alg}}] \sim n \)

\( \# M_{\text{opt}} = 2n \)
What to do?

RANKING: works as follows.

At time $\mathcal{P}$, choose random permutation of $L$.

At time $t > \mathcal{P}$ when voter $t \in R$ arrives, match to the unmatched neighbor that occurs earliest in the random permutation.