$\boxed{\text{8 Sep 2021}}$

① Fractional matching LP and its dual

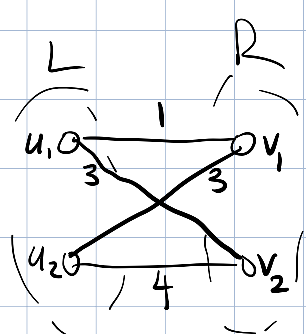
② Introduce online matching

$$\text{minimize} \sum_{\text{edges } (u,v)} c(u,v)\, x_{uv}$$

$$\text{subject to} \quad \sum_{v} x_{uv} = 1 \qquad \forall u \in L$$

$$\sum_{u} x_{uv} = 1 \qquad \forall v \in R$$

$$x_{uv} \in \{0,1\} \qquad \forall (u,v) \in E$$



Instead of $x_{u_1 v_2}$ I'll write x_{12}.

$$\text{min.} \quad x_{11} + 3x_{12} + 3x_{21} + 4x_{22}$$

$$\text{s.t.} \quad x_{11} + x_{12} = 1$$

$$x_{21} + x_{22} = 1$$

$$x_{12} + x_{22} = 1$$

$$x_{11} + x_{21} = 1$$

$$x_{uv} \in \{0,1\} \quad \forall u,v.$$

$$2x_{11} + 2x_{21} = 2$$
$$-x_{11} - x_{12} = -1$$
$$4x_{12} + 4x_{22} = 4$$

$$x_{11} + 3x_{12} + 2x_{21} + 4x_{22} = 5$$

$$x_{21} \geq 0$$

$$x_{11} + 3x_{12} + 3x_{21} + 4x_{22} \geq 5$$

In general...

$$\begin{aligned} \min \quad & \sum c(u,v) \cdot x_{uv} & \\ \text{s.t.} \quad & \sum_v x_{uv} = 1 & \forall u \in L \\ & \sum_u x_{uv} = 1 & \forall v \in R \\ & x_{uv} \geq 0 & \forall (u,v) \end{aligned}$$

scale by $y_u \in \mathbb{R}$

scale by $y_v \in \mathbb{R}$

$$\sum_v y_u x_{uv} = y_u \qquad \forall u \in L$$
$$\sum_u y_v x_{uv} = y_v \qquad \forall v \in R$$

$$\sum_{(u,v)\in E} (y_u + y_v)\, x_{uv} = \sum_{u\in L} y_u + \sum_{v\in R} y_v$$

if $y_u + y_v \leq c(u,v)$ $\forall$ edge (u,v),
the following deduction is justified...

$$\sum_{(u,v)\in E} c(u,v)\cdot x_{uv} \geq \sum_{u\in L} y_u + \sum_{v\in R} y_v .$$

The last line is an equation (not inequality) provided that the only pairs (u,v) for which $y_u + y_v < c(u,v)$ are those for which $x_{uv} = 0$. Equivalently...

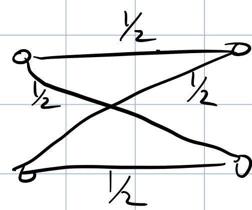
"complementary slackness" When $x_{uv} > 0$, $y_u + y_v = c(u,v)$.

Def. A fractional perfect matching $\vec{x}$ is a solution of

$$\begin{cases} \sum_v x_{uv} = 1 & \forall u \in L \\ \sum_u x_{uv} = 1 & \forall v \in R \end{cases}$$

$$x_{uv} \geq 0$$

E.g.



Fractional perfect matching of an $n \times n$ complete bipartite graph form a convex polyhedron.

<u>Birkhoff–von Neumann Theorem</u> Every fractional perfect matching in a bipartite graph is a convex combination of perfect matchings.

"a.k.a. "weighted average"

$\sum w_i \vec{x}_i$

where $w_i \geq 0 \ \forall i$

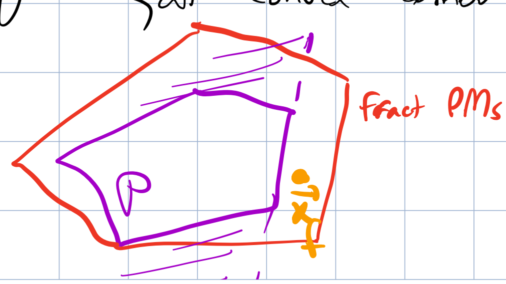
$\sum w_i = 1$

<u>Proof</u>.

By contradiction.

Suppose $\vec{x}_f$ is a FPM but not a convex combo of PMs.

Let $P = \{$all convex combo of PMs$\}$.



$\exists$!

There exists a linear inequality $\vec{c}\cdot\vec{x} \geq b$
that is satisfied by every PM.
violated by $\vec{x_f}$. $\vec{c}\cdot\vec{x_f} < b$.

Consider the min-cost perfect matching problem

$$\min \sum c_{uv} x_{uv} \qquad (\text{a.k.a } \vec{c}\cdot\vec{x})$$

$$\text{subj to} \left\{ \begin{array}{ll} \sum_v x_{uv} = 1 & \forall u \\ \sum_u x_{uv} = 1 & \forall v \\ x_{uv} \geq 0 & \end{array} \right\} \text{ a.k.a. } \vec{x} \text{ is a FPM.}$$

When we run last week's algo we
get an <u>actual</u> perf matching M
(with assoc vector $\vec{x_M}$) and
M-compatible labeling $\vec{y}$.

The labeling $\vec{y}$ can be used as
scale-factors on equations $\sum_u x_{uv} = 1$, $\sum_v x_{uv} = 1$,
to certify that $\vec{c}\cdot\vec{x} \geq \left(\sum_u y_u + \sum_v y_v\right)$
$\forall$ FPM $\vec{x}$.

M-compatibility means

$$b \leq \vec{c} \cdot \vec{x_n} = \sum_u y_u + \sum_v y_v.$$

$$b > \vec{c} \cdot \vec{x_f} \geq \sum_u y_u + \sum_v y_v$$

contradiction!