3 Sept 2021 — The Fractional Matching LP and Its Dual

RECAP

Def. If M matching, y labeling, we call y <u>M-compatible</u> if

1. $\forall e=(u,v)$ $\quad c(u,v) \geq y_u + y_v$
2. $\forall e=(u,v) \in M$ $\quad c(u,v) = y_u + y_v$
3. $\forall u \in L \cap F$ $\quad y_u = \max\{ y_w \mid w \in L \}$.
4. $\forall v \in R \cap F$ $\quad y_v = \max\{ y_w \mid w \in R \}$.

NEW ALGORITHM

$M_0 = \emptyset \qquad \vec{y} = \vec{0}$

for $k = 1, 2, \ldots, \frac{n}{2}$

compute reduced costs $c^y(u,v)$ in G_M.

use Dijkstra's algorithm to find

(i) path P from $L \cap F$ to $R \cap F$ of minimum reduced cost.

(ii) a label d_v for each $v \in V(G_M)$ representing shortest path cost from $L \cap F$ to v.

($d_v = \infty$ if no path exists)

$$M_{k+1} = M_k \oplus P$$ $x^+ = \max\{x, 0\}$.

$y_u \leftarrow y_u + (c^y(P) - d_u)^+$ for all $u \in L$

$y_v \leftarrow y_v - (c^y(P) - d_v)^+$ for all $v \in R$.

end for

output $M_{n/2}$.

Lemma. Assume $c(u,v) \geq 0$ $\forall$ edge (u,v), after every loop iteration $\vec{y}$ is compatible with M_k.

Proof. Induction on k.

When $k = 0$, $M_k = \emptyset$ $\vec{y} = \vec{0}$.

All 4 properties hold.

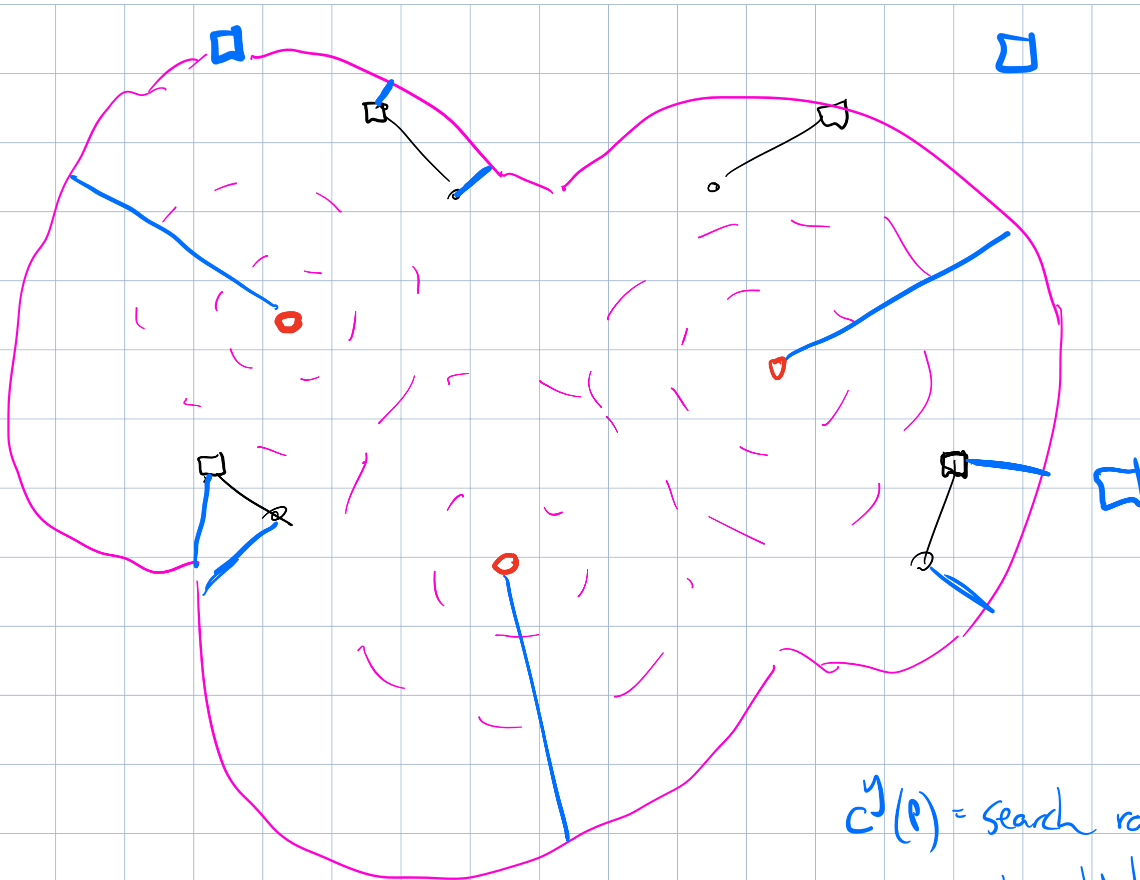
Assuming M_k compatible with $\vec{y}$,

let $M = M_k$

$M' = M_{k+1}$

$y' = \vec{y}$ after update in loop iteration k.

We aim to show y' is M'-compatible.



$c^y(P)$ = search radius at which we meet $R \cap F$

d_u, d_v = distance from "seed set" $L \cap F$.

$(c^y(P) - d_u)^+$ = distance from boundary.

To prove: $\forall\, e = (u,v)$

$$c(u,v) \geq y'_u + y'_v$$

Equality holds when $e \in M'$.

Case 1. $e \in M$.

$$c(u,v) = y_u + y_v \quad \text{(ind hyp, prop 2)}$$

$$y'_u = y_u + (c^y(P) - d_u)^+$$

$$y'_v = y_v - (c^y(P) - d_v)^+$$

In G_M, u has only 1 incoming edge, namely (v,u).
Cost of that edge, $c^y(v,u)$, was defined as

$$c^y(v,u) = y_u + y_v - c(u,v) = 0.$$

So

$$d_u = d_v. \Rightarrow (c^y(P) - d_u)^+ = (c^y(P) - d_v)^+$$

$$\Rightarrow y'_u + y'_v = y_u + y_v.$$

Case 2 & 3. $e \notin M$. If G_M e is oriented $u \to v$.

$$c^y(u,v) = c(u,v) - y_u - y_v.$$

$$d_v \leq d_u + c(u,v) - y_u - y_v$$

$$(y_u - d_u) + (y_v + d_v) \leq c(u,v)$$

... and the two sides are equal iff the shortest path $L\cap F$ to v happens to go through (u,v).

Case 2. $e \in P \setminus M$.

Then criterion for equality above holds.

$$(y_u - d_u) + (y_v + d_v) \leq c(u,v)$$

Also $d_u, d_v \leq c^y(P)$.

$$c^y(P) - d_u \geq 0$$
$$c^y(P) - d_v \geq 0.$$

$$y'_u = y_u + c^y(P) - d_u$$
$$y'_v = y_v - c^y(P) + d_v.$$

$$y'_u + y'_v = (y_u - d_u) + (y_v + d_v) = c(u,v).$$

Case 3. $e \notin M \cup P$.

$$(u,v) \in E(G_M) \Rightarrow \quad d_v \leq d_u + c^y(u,v)$$
$$c^y(P) - d_u \leq c^y(P) - d_v + c^y(u,v)$$
$$(c^y(P) - d_u)^+ \leq (c^y(P) - d_v)^+ + c^y(u,v)$$

$$y_u' + y_v' = y_u + (c^y(P) - d_u)^+ + y_v - (c^y(P) - d_v)^+$$
$$\leq y_u + \cancel{(c^y(P) - d_v)^+} + c^y(u,v) + y_v - \cancel{(c^y(P) - d_v)^+}$$
$$= y_u + c^y(u,v) + y_v$$
$$= c(u,v)$$

Props 1,2 now verified.

Props 3,4... go back and look at the update rule.

$$y_u' = y_u + (c^y(P) - d_u)^+$$

$$y_v' = y_v - (c^y(P) - d_v)^+$$

Min cost matching as a linear optimization problem.....

For a matching M define a vector $\vec{x}$ by
$$x_{uv} = \begin{cases} 1 & \text{if } (u,v) \in M \\ 0 & \text{if } (u,v) \notin M. \end{cases}$$

Min-cost perfect matching can be formulated as

$$\text{Minimize} \sum_{(u,v)\in E} c(u,v)\cdot x_{uv}$$

subj to

$$\sum_{v\in R} x_{uv} = 1 \qquad \forall u\in L$$

$$\sum_{u\in L} x_{uv} = 1 \qquad \forall v\in R$$

$$x_{uv} \in \{0,1\}.$$