1 Sep 2021 — Primal–Dual Min-Cost Bipartite Matching

RECAP

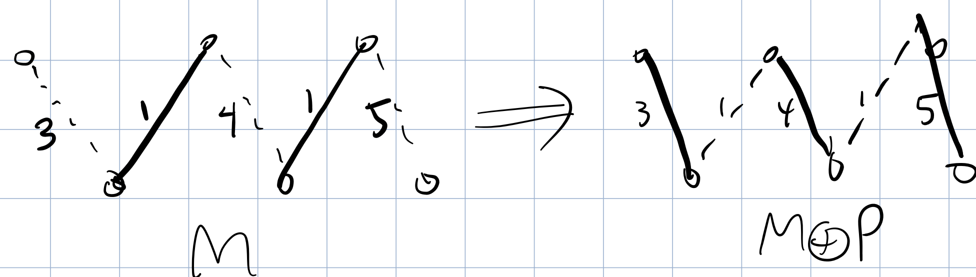
G bipartite $(L \cup R, E)$

M matching

F {free vertices} = {vertices that don't belong to any edge in M}

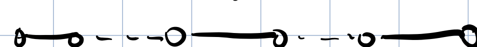
G_M directed graph with $e \in M$ oriented $R \to L$, $e \notin M$ $L \to R$

P



$\text{cost}(M) = 1+1 \qquad \text{cost}(M \oplus P) = 3+4+5$

$$\Delta c(P; M) = \text{cost}(M \oplus P) - \text{cost}(M)$$
$$= 3 - 1 + 4 - 1 + 5$$



$$\text{cost}(u,v) = \begin{cases} c(u,v) & \text{if } (u,v) \notin M \\ -c(u,v) & \text{if } (v,u) \in M \end{cases}$$

GREEDY ALGORITHM

$M_0 \leftarrow \emptyset$

for $k = 0, \ldots, \frac{n}{2} - 1$

compute G_M with its edge costs defined as above.

let $\boxed{P_k = \text{min cost directed path from } L \cap F \text{ to } R \cap F}$

let $M_{k+1} = M_k \oplus P_k$

end for

output $M_{n/2}$.

efficient search procedure?

Bellman Ford

time $O(nm)$ per iter

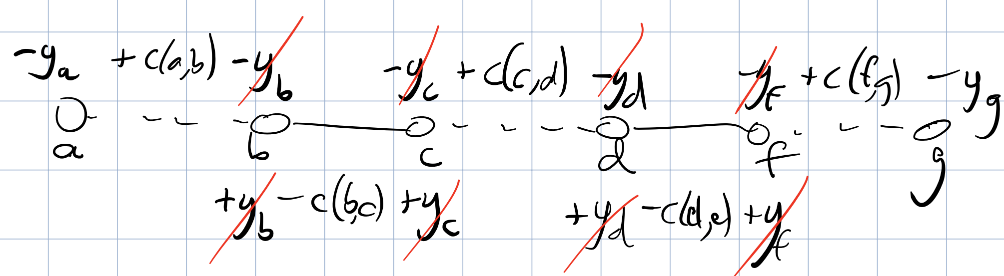
iterations $n/2$

Algorithm takes $O(mn^2)$ overall.

Along with M_k we will be computing a labeling $\vec{y}$ that assigns a number (pos or neg) to each vertex.

Def 1. If M matching, y labeling, the edges of G_M have <u>reduced costs</u>

$$c^y(u,v) = \begin{cases} c(u,v) - y_u - y_v & \text{if } (u,v) \notin M \\ y_u + y_v - c(u,v) & \text{if } (u,v) \in M. \end{cases}$$



$$c^y(P) = \Delta c(P;\, M) - y_a - y_g$$

Def. 2 If M matching, y labeling, we call y <u>M-compatible</u> if

1. $\forall e=(u,v)$ $\quad c(u,v) \geq y_u + y_v$
2. $\forall e=(u,v) \in M$ $\quad c(u,v) = y_u + y_v$
3. $\forall u \in L \cap F$ $\quad y_u = \max\{ y_w \mid w \in L\}$.
4. $\forall v \in R \cap F$ $\quad y_v = \max\{ y_w \mid w \in R\}$.

Obs. If M is a matching, y is M-compat,
$c^y(u,v) \geq 0 \quad \forall$ edge $(u,v) \in G_M$.

We can use Dijkstra to find P
that minimizes $c^y(P)$.
The same path minimizes $\Delta c(P; M)$.
because if P is a path from $s \in L \cap F$ to $t \in R \cap F$

$$\Delta c(P;M) = c^y(P) + y_s + y_t$$

By properties 3 & 4 this quantity only depends on the fact $s,t \in F$ not on which free vertices they are.

Lemma - If M is a matching and y is M-compatible then M has the min cost among all matching with same # edges.

Proof. Let M, M' be 2 matching of same size, y an M-compatible labeling.

Let $W(M) = \{\text{vertices matched in } M\}$
$W(M') = \{ \text{---} \text{ " } \text{------} \ M' \}$.

Then

$$c(M) = \sum_{(u,v)\in M} c(u,v) = \sum_{(u,v)\in M} (y_u + y_v) = \sum_{w\in W(M)} y_w$$

$$c(M') = \sum_{(u,v)\in M'} c(u,v) \geq \sum_{(u,v)\in M'} (y_u + y_v) = \sum_{w\in W(M')} y_{w'}$$

$$c(M') - c(M) \geq \sum_{w\in W(M')} y_{w'} - \sum_{w\in W(M)} y_w .$$

$$\geq 0 \quad \text{by props } 3,4.$$

NEW ALGORITHM

$M_0 = \emptyset \qquad \vec{y} = \vec{0}$

for $k = 1, 2, \ldots, \frac{n}{2}$

compute reduced costs $c^y(u,v)$ in G_M.

use Dijkstra's algorithm to find

(i) path P from LnF to RnF of minimum reduced cost.

(ii) a label d_v for each $v \in V(G_M)$ represting shortest path cost from LnF to v.

($d_v = \infty$ if no path exists)

$$M_{k+1} = M_k \oplus P$$

$x^+ = \max\{x, 0\}$.

$y_u \leftarrow y_u + (c^y(P) - d_u)^+$ for all $u \in L$

$y_v \leftarrow y_v - (c^y(P) - d_v)^+$ for all $v \in R$.

end for

output $M_{n/2}$.

Lemma. At start of each iteration, assuming $c(u,v) \geq 0$ $\forall$ edge (u,v), $\vec{y}$ is M_k-compatible.