CS 6817: Special Topics in Complexity Theory **Spring 2025**

Lecture 16 & 17

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1 KKL Theorem

In this lecture we return to the study of influence in Boolean functions.

1.1 Background

We consider Boolean functions $f : \{-1,1\}^n \to \{-1,1\}$. Recall from previous lectures that the influence of the i-th variable on function f is defined as

$$I_i(f) = \Pr_{x\sim\{-1,1\}^n}[f(x) \neq f(x^{\oplus i})] = \|D_i f\|_2^2$$

where $x^{\oplus i}$ represents the input x with the i-th bit flipped, and $D_i f$ is the discrete derivative in the i-th direction.

$$D_i f(x) = \frac{f(x^{i\to 1}) - f(x^{i\to -1})}{2}$$

We define the total influence of a function is the sum of the influences of all variables,

$$I(f) = \sum_{i=1}^{n} I_i(f)$$

1.2 Maximum Influence

An important question we have is: what can we say about the maximum influence of any variable in a function? What is a good lower bound for $\max_i\{I_i(f)\}$?

In this lecture we focus on balanced functions (they are functions where $\Pr[f(x) = 1] = \Pr[f(x) = -1] = \frac{1}{2}$ when x is chosen uniformly at random).

From previous lectures (lecture 5), we established the Poincaré Inequality,

$$I(f) \geq \mathrm{Var}(f)$$

and for balanced Boolean functions, $\mathrm{Var}(f) = 1$, so we have $I(f) = \Omega(1)$. This gives us a simple lower bound on the maximum influence

$$\max_i\{I_i(f)\} \geq \frac{I(f)}{n} = \Omega\left(\frac{1}{n}\right)$$

The KKL Theorem provides a stronger result.

1.3 The KKL Theorem

This theoreom was given by Kahn, Kalai, and Linial in 1988.

Theorem 1.1: For any Boolean function $f : \{-1, 1\}^n \to \{-1, 1\}$, there exists a variable $i \in [n]$ and constant $c > 0$ such that

$$I_i(f) \geq c \cdot \mathrm{Var}(f) \cdot \frac{\log n}{n}$$

This result is improves the $\Omega(1/n)$ bound by a logarithmic factor.

From lecture 15, recall

Claim 1.2: For any $g : \{-1, 1\}^n \to \{-1, 0, 1\}^n$,

$$Stab_{\frac{1}{3}}[g] \leq \|g\|_2^3$$

The proof for this claim is as follows. We begin by applying Holder's inequality with $\frac{1}{4}$ and $\frac{3}{4}$,

$$\begin{aligned}\mathrm{Stab}_{\frac{1}{3}}[g] &= \langle T_{\frac{1}{3}} g, g\rangle \\ &= \|T_{\frac{1}{2}} g\|_4 \|g\|_{\frac{4}{3}}\end{aligned}$$

Alternatively we can express it as

$$\begin{aligned}\langle T_{\frac{1}{3}} g, g\rangle &= \langle T_{\frac{1}{\sqrt{3}}} g, T_{\frac{1}{\sqrt{3}}} g\rangle \\ &= \|T_{\frac{1}{\sqrt{3}}} g\|_2^2\end{aligned}$$

and

$$\begin{aligned}\|T_{\frac{1}{2}} g\|_4 \|g\|_{\frac{4}{3}} &= \|T_{\frac{1}{\sqrt{3}}} (T_{\frac{1}{\sqrt{3}}} g)\|_4 \|g\|_{\frac{4}{3}} \\ &\leq \|T_{\frac{1}{3}} (T_{\frac{1}{3}} g)\|_4 \|g\|_{\frac{4}{3}}\end{aligned}$$

When applying the noise parameter δ to obtain the 4-norm, this is at most the 2-norm of the function.

$$\|T_{\frac{1}{3}} (T_{\frac{1}{3}} g)\|_4 \|g\|_{\frac{4}{3}} \leq \|T_{\frac{1}{\sqrt{3}}} g\|_2 \|g\|_{\frac{4}{3}}$$

Theorem 1.3: $(2, 4)$ **Hypercontractivity** For any real-valued Boolean function f

$$\|T_\rho f\|_4 \leq \|f\|_2$$

Applying this theorem to our context implies

$$\begin{aligned}\|T_{\frac{1}{\sqrt{3}}} g\|_2 &\leq \|g\|_{\frac{4}{3}} \\ &= \mathbb{E}[|g(x)|]^{\frac{3}{4}} && \text{(by definition)} \\ &= \mathbb{E}[|g(x)|^2]^{\frac{1}{2} \cdot \frac{3}{4}} && \text{(for squared functions the absolute value becomes redundant)} \\ &= \|g\|_2^{\frac{3}{2}}\end{aligned}$$

1.4 Proof for the KKL Theorem

Corollary 1.4: For any Boolean function $f : \{-1,1\}^n \to \{-1,1\}$ and any index $i \in [n]$:

$$Stab_{\frac{1}{3}}[D_i f] \leq \|D_i f\|_2^3 = I_i(f)^{\frac{1}{3}}$$

This corollary tells us that when examining the noise stability of the discrete derivative of any function, we can upper bound it by the influence raised to a power greater than 1. Assuming $\mathrm{Var}(f)$ is constant, we can prove that the influence of each variable is at least $\Omega\left(\frac{\log n}{n}\right)$.

Remark 1.5: If the total influence $I(f) = \Omega(\log n)$, then the KKL theorem follows trivially by the pigeonhole principle.

Claim 1.6: There exists a constant c such that for a Boolean function f

$$\exists\, i \text{ such that } I_i(f) \geq 2^{-c \cdot I}$$

where $I = I(f)$.

From corollary 1.4, we can write

$$\begin{aligned}\sum_{i=1}^{n} Stab_{\frac{1}{3}}[D_i f] &\leq \sum_{i=1}^{n} I_i(f)^{\frac{3}{2}} \\ &\leq \max_i \{I_i(f)\}^{\frac{1}{2}} \cdot \sum_{i=1}^{n} I_i(f) \\ &= \max_i \{I_i(f)\}^{\frac{1}{2}} \cdot I(f)\end{aligned}$$

A Fourier formula for $\mathrm{Stab}_{1/3}[D_i f]$ is

$$= \langle T_{\frac{1}{3}} D_i f, D_i f \rangle \qquad \text{(by definition)}$$

$$= \left\langle T_{\frac{1}{3}} \left(\sum_{i \in S} \hat{f}(S) \chi_{S \setminus \{i\}} \right), \sum_{i \in S} \hat{f}(S) \chi_{S \setminus \{i\}} \right\rangle \qquad \text{(any monomial without } x_i \text{ vanishes in this calculation)}$$

$$= \sum_{i \in S} \left(\frac{1}{3}\right)^{|S|-1} \hat{f}(S)^2$$

Summing over all variables

$$\begin{aligned}\sum_{i=1}^{n} Stab_{\frac{1}{3}}[D_i f] &= \sum_{i=1}^{n} \sum_{i \in S} \left(\frac{1}{3}\right)^{|S|-1} \hat{f}(S)^2 \\ &= \sum_{S \subseteq [n]} \left(\frac{1}{3}\right)^{|S|-1} |S| \hat{f}(S)^2 \\ &\geq \frac{1}{3} \mathbb{E}_{S \sim S_f} \left[\frac{1}{3}^{|S|} \right] \\ &\geq \frac{1}{3} \cdot 3^{-\mathbb{E}[|S|]} \quad \text{(by convexity)}\end{aligned}$$

This gives us a lower bound related to the influence. Therefore we get

$$2^{-c \cdot I(f)} \leq \max_i \{I_i(f)\}^{\frac{1}{2}} \cdot I(f)$$
$$\Rightarrow \max_i \{I_i(f)\} \geq \frac{2^{-2c_i \cdot I(f)}}{I(f)^2}$$

This result is also known as the edge-isoperimetric version of KKL.
To obtain the standard KKL theorem from this, we consider two cases:

- **Case 1:** $I(f) = \beta \log n$ for some constant β

$$\Rightarrow \exists\ i \text{ such that } I_i(f) \geq \frac{\beta \log n}{n}$$

  This immediately gives us the desired result.

- **Case 2:** $I(f) < \beta \log n$ Using theorem 1.1 and corollary 1.4:

$$\exists\ i \text{ such that } I_i(f) \geq 2^{-c\beta \log n}$$

We want to choose β such that $c\beta < 1$, so let us choose $\beta = \frac{0.9}{c}$. This gives us

$$\exists\ i \text{ such that } I_i(f) \geq n^{-0.9}$$

For sufficiently large n, this bound is stronger than the required $\Omega\left(\frac{\log n}{n}\right)$, completing the proof of the KKL theorem.

2 Freidgut Junta Theorem

What can we say more when $I(f)$ is small, particularly, when $I(f) << \log n$?

2.1 Background

Theorem 2.1: For any Boolean function $f : \{-1, 1\}^n \to \{-1, 1\}$ and any $0 \leq \epsilon \leq 1$, the function f is ϵ-close to a k-junta where

$$k = 2^{\frac{c \cdot I(f)}{\epsilon}}$$

Here, ϵ-close means that f differs from the junta on at most an ϵ fraction of inputs, and $I(f)$ is the total influence of f.

Definition 2.2: A function $g : \{-1, 1\}^n \to \mathbb{R}$ is a k-junta if there exists a set $S \subseteq [n]$ with $|S| \leq k$ and a function $h : \{-1, 1\}^S \to \mathbb{R}$ such that for all x,

$$g(x) = h(x_S)$$

In the case 2 considered for the KKL theorem, if $I(f)$ is small we can pick a coordinate with influence. Theorem 2.1 shows that we can keep collecting these influential coordinates to approximate f by a function that depends only on these coordinates.

2.2 Proof for the Junta Theorem

The proof relies on a low-degree version of hypercontractivity.

Definition 2.3: (Low-degree Hypercontractivity) For any real-valued function $f : \{-1,1\}^n \to \mathbb{R}$ with $\deg(f) \leq d$

$$\|f\|_4 \leq (3)^{d/2}\|f\|_2$$

Since the total influence $I(f)$ is small, there can only be a limited number of variables with significant influence. Let

$$J = \{i \cdot I_i(f) \geq \delta\}$$

be the set of coordinates with influence at least $\delta = 2^{-\frac{C \cdot I(f)}{\epsilon}}$.

Define g as

$$g(x) = \sum_{S \subseteq J, |S| \leq d} \hat{f}(S)\chi_S(x)$$

And

$$h(x) = \text{sign}(g(x)).$$

By construction, g, h is are $|J|$-juntas.

Definition 2.4: (Degree-d truncation) For a function $g : \{-1,1\}^n \to \{-1,1\}$ that can be written as

$$g(x) = \sum_{S \subseteq [n]} c_S \chi(x)^S,$$

for $0 \leq d \leq n$, we can define

$$g^{\leq d}(x) = \sum_{S \subseteq [n], |S| \leq d} c_S \chi(x)^S.$$

In this case, we choose

$$d = \frac{10 \cdot I(f)}{\epsilon}$$

Claim 2.4: The set J cannot be too large,

$$|J| \leq \frac{I(f)}{\delta}$$

We need to show that $\|f - g\|_2^2$ is small, which will establish that f is ϵ-close to a junta (junta g).

$$\|f - g\|_2^2 = \sum_{|S| > d} \hat{f}(S)^2 + \sum_{S \not\subseteq J, |S| \leq d} \hat{f}(S)^2$$

The first term is small because of the Fourier concentration (the Markov bound on the spectral sample)

$$\sum_{|S| > \frac{10 \cdot I(f)}{\varepsilon}} \hat{f}(S)^2 \leq \frac{\varepsilon}{10}$$

For the second term,

$$\sum_{S \not\subseteq J, |S| \leq d} \hat{f}(S)^2 \leq \sum_{i \notin J} \sum_{i \in S, |S| \leq d} \hat{f}(S)^2$$
$$\leq \sum_{i \notin J} \|D_i f^{\leq d}\|_2^2$$

where $D_i f^{\leq d}$ is the degree-d truncation of the discrete derivative,

$$D_i f^{\leq d} = \sum_{i \in S, |S| \leq d} \hat{f}(S) \chi_{S \setminus \{i\}}(x).$$

To rewrite this bound, we use degree bounded hypercontractivity.

Claim 2.5 (degree bounded hypercontractivity): Let $g : \{-1, 1\}^n \to \{-1, 0, 1\}$. For any d,

$$\left|\left|g^{\leq d}\right|\right|_2^2 \leq \sqrt{3}^d \cdot ||g||_2^{\frac{5}{2}}.$$

Proof: By definition, we have

$$\left|\left|g^{\leq d}\right|\right|_2^2 = \langle g^{\leq d}, g^{\leq d} \rangle.$$

By orthormality, we can say

$$\left|\left|g^{\leq d}\right|\right|_2^2 = \langle g^{\leq d}, g^{\leq d} \rangle = \langle g^{\leq d}, g \rangle.$$

This gives an ideal form to apply Hölder's inequality with $p = \frac{1}{4}, q = \frac{3}{4}$, to get

$$\left|\left|g^{\leq d}\right|\right|_2^2 = \langle g^{\leq d}, g \rangle$$
$$\leq \left|\left|g^{\leq d}\right|\right|_4 ||g||_{\frac{4}{3}}.$$

Substituting using the 2-$\frac{4}{3}$ hypercontractivity, we get

$$\left|\left|g^{\leq d}\right|\right|_2^2 \leq \left|\left|g^{\leq d}\right|\right|_4 ||g||_{\frac{4}{3}}$$
$$\leq 3^{\frac{d}{2}} ||g||_2 ||g||_{\frac{4}{3}}.$$

For the last step, we note that by the definition of norm, we see

$$||g||_{\frac{4}{3}} = \left(\mathrm{E}\left[|g|^{\frac{4}{3}}\right]\right)^{\frac{3}{4}} = \left(\mathrm{E}[|g|^2]^{\frac{1}{2}}\right)^{\frac{3}{2}} = (\mathrm{E}[|g|])^{\frac{3}{2}} = ||g||_2^{\frac{3}{2}}.$$

Substituting, we get

$$\left|\left|g^{\leq d}\right|\right|_2^2 \leq 3^{\frac{d}{2}} ||g||_2 ||g||_{\frac{4}{3}}$$
$$\leq 3^{\frac{d}{2}} ||g||_2 ||g||_2^{\frac{3}{2}} = 3^{\frac{d}{2}} |g||_2^{\frac{5}{2}}.$$

Using this claim, we see

$$\begin{aligned}\|D_i f^{\leq d}\|_2^2 &= \langle D_i f^{\leq d}, D_i f^{\leq d}\rangle \\ &= \langle D_i f^{\leq d}, D_i f\rangle \\ &\leq \|D_i f^{\leq d}\|_4 \|D_i f\|_{\frac{4}{3}} \\ &\leq (3)^{d/2} \|D_i f\|_2^{5/2} \quad \text{(by hypercontractivity)}\end{aligned}$$

Putting it all together,

$$\sum_{S \not\subseteq J, |S| \leq d} \hat{f}(S)^2 \leq 3^{d/2} \sum_{i \notin J} I_i(f) \cdot \sqrt{I_i(f)}$$

3 Talagrand's Version of KKL, an Improved KKL

When a function has a low influence, the majority of the coefficients must be comparable to $\frac{\log n}{n}$. Talagrand's Version of KKL is stated as follows:

Theorem 3.1. *For any $f : \{-1, 1\}^n \to \{-1, 1\}$,*

$$\sum_{i=1}^{n} \frac{I_i(f)}{\log\left(\frac{1}{I_i(f)}\right)} \geq \mathit{Var}(f).$$

Proof: By definition, we can say

$$\mathrm{Var}(f) = \sum_{S \neq \emptyset} \hat{f}(S)^2.$$

Expanding by each $i \in [n]$, we get

$$\mathrm{Var}(f) = \sum_{S \neq \emptyset} \hat{f}(S)^2 = \sum_{i=1}^{n} \sum_{S \ni i} \frac{1}{|S|} \hat{f}(S)^2.$$

For each i, we define

$$g_i(x) = \sum_{S \ni i} \frac{1}{|S|^{\frac{1}{2}}} \hat{f}(S) \chi_S(x).$$

Therefore, we can rewrite the variance as

$$\mathrm{Var}(f) = \sum_{i=1}^{n} \sum_{S \ni i} \frac{1}{|S|} \hat{f}(S)^2 = \sum_{i=1}^{n} ||g_i||_2^2.$$

Since $||g_i||_2^2 = \sum_{S \ni i} \frac{1}{|S|} \hat{f}(S)^2$, we can expand each $||g_i||_2^2$ as

$$||g_i||_2^2 = \sum_{|S| \leq d_i, S \ni i} \frac{1}{|S|} \hat{f}(S)^2 + \sum_{|S| > d_i, S \ni i} \frac{1}{|S|} \hat{f}(S)^2$$

where

$$d_i = C \cdot \log\left(\frac{1}{I_i(f)}\right).$$

Since $\frac{1}{|S|} \leq 1$ for $|S| \leq d_i$ and $\frac{1}{|S|} \leq \frac{1}{d_i}$ for $|S| \leq d_i$, we can say

$$||g_i||_2^2 = \sum_{|S| \leq d_i, S \ni i} \frac{1}{|S|} \hat{f}(S)^2 + \sum_{|S| > d_i, S \ni i} \frac{1}{|S|} \hat{f}(S)^2$$

$$\leq ||g_i||_2^2 = \sum_{|S| \leq d_i, S \ni i} \hat{f}(S)^2 + \frac{1}{d_i} \sum_{|S| > d_i, S \ni i} \hat{f}(S)^2.$$

We bound each sum separately. $\sum_{|S|>d_i, S\ni i} \hat{f}(S)^2$ is bounded above by $\frac{\epsilon}{10}$ by Markov's inequality. Since this holds for any $\epsilon > 0$, this term effectively disappears from our sum. Meanwhile, we see that by the definition of the derivative,

$$\sum_{|S| \leq d_i, S \ni i} \frac{1}{|S|} \hat{f}(S)^2 = \left|\left| D_i f^{\leq d} \right|\right|_2^2.$$

By the degree bound version of hypercontractivity and the definition of influence, we can say

$$\sum_{|S| \leq d_i, S \ni i} \frac{1}{|S|} \hat{f}(S)^2 = \left|\left| D_i f^{\leq d_i} \right|\right|_2^2 \leq 3^{\frac{d_i}{2}} \, ||D_i f||_{\frac{5}{2}}^2 = I_i(f)^{\frac{5}{4}} \cdot 3^{\frac{d_i}{2}}.$$

Returning to the equation with the variance, we get

$$\mathrm{Var}(f) = \sum_{i=1}^n ||g_i||_2^2$$

$$\leq \sum_{i=1}^n I_i(f)^{\frac{5}{4}} \cdot 3^{\frac{d_i}{2}}$$

$$\leq \sum_{i=1}^n I_i(f) \cdot 3^{\frac{d_i}{2}}$$

since the influence of each coordinate is a probability between 0 and 1. Substituting the value of d_i, we get

$$\mathrm{Var}(f) = \sum_{i=1}^n ||g_i||_2^2$$

$$\leq \sum_{i=1}^n I_i(f) \cdot 3^{\frac{d_i}{2}}$$

$$= \sum_{i=1}^n I_i(f) \cdot 3^{\frac{C \cdot \log\left(\frac{1}{I_i(f)}\right)}{2}}$$

$$= \sum_{i=1}^n \frac{I_i(f)}{\log\left(\frac{1}{I_i(f)}\right)}$$

by setting C according so that any extra constants cancel out.

4 Freidgut-Kalai-Naor (FKN) Theorem

We saw before in class that if a Boolean valued function has degree 1 (i.e., all its Fourier mass is on level 1), then it must be a dictator (or anti-dictator). The FKN theorem, stated below, proves a robust version of this fact.

Theorem 4.1. *Let $f : \{-1, 1\}^n \to \{-1, 1\}$ be such that*

$$W^1[f] = \sum_{i=1}^{n} \hat{f}(\{i\})^2 \geq 1 - \epsilon.$$

Then, f is ϵ-close to x_i or $-x_i$.

Proof: We define

$$h = \sum_{i=1}^{n} \hat{f}(\{i\}) x_i.$$

For ease of notation, we will say $\hat{f}(i) = \hat{f}(\{i\})$. Squaring both sides, we see

$$h^2 = \sum_{i=1}^{n} \hat{f}(i)^2 + 2 \sum_{i<j}^{n} \hat{f}(i)\hat{f}(j) x_i x_j.$$

Since $\mathrm{Var}(h^2) = \sum_{S \neq \emptyset} \hat{h}(S)^2$ by definition, we can use the above equation to say

$$\mathrm{Var}(h^2) = 4 \sum_{i<j}^{n} \hat{f}(i)^2 \hat{f}(j)^2$$

$$= 2 \left(\left(\sum_{i=1}^{n} \hat{f}(i)^2 \right)^2 - \sum_{i=1}^{n} \hat{f}(i)^4 \right)$$

$$\geq 2 \left((1 - \epsilon)^2 - \max_i \{\hat{f}(i)^2\} \right).$$

Note that the last inequality comes from the assumption on f. Rearranging the inequality, we get

$$\max_i \{\hat{f}(i)^2\} \geq (1 - \epsilon)^2 - \frac{\mathrm{Var}(h^2)}{2}.$$

Next, we will bound $\mathrm{Var}(h^2)$ to demonstrate that it is small. To do this, we define

$$g(x) = \sum_{i<j} \hat{f}(i)\hat{f}(j) x_i x_j,$$

which has the nice property that

$$||g||_2^2 = C \cdot \mathrm{Var}(h^2).$$

We will make of the fact that $||g||_2^2 \leq 3^{\deg(g)} ||g||_1$. In this case, it allows us to say that

$$C \cdot \mathrm{Var}(h^2) = ||g||_2^2 \leq 3^{\deg(g)} ||g||_1$$

$$\mathrm{E}[h^2] = 1$$

$$\sum_{S \subseteq [n]} \hat{h}(S) = 1$$

$$\hat{h}(\emptyset) + \sum_{S \neq \emptyset} \hat{h}(S) = 1$$

$$\hat{h}(\emptyset) + \mathrm{Var}(h) = 1.$$

Since $\hat{h}(\emptyset) = 1$, this allows us to see

$$\mathrm{Var}(h) = 0.$$

As a result, our inequality simplifies to

$$\max_i \{\hat{f}(i)^2\} \geq (1 - \epsilon)^2$$

$$\max_i \{|\hat{f}(i)|\} \geq 1 - \epsilon$$

$$1 - \max_i \{|\hat{f}(i)|\} \leq \epsilon.$$

This inequality shows that for $i' \in [n]$ which maximizes $\max_i \{|\hat{f}(i)|\}$, f is ϵ-close to $x_{i'}$ or $-x_i$ (whichever ends up being 1).