CS 6817: Special Topics in Complexity Theory **Spring 2025**

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Lecturer: Eshan Chattopadhyay *Scribe: Ruijia Cao*

1 Recap from Last Lecture

In the previous lecture, we considered the question of sharpness of the Poincaré inequality: does there exist a boolean-valued function $f\colon \{\pm 1\}^n \to \{\pm 1\}$ such that $\max_i \mathsf{I}_i(f) = O(1/n)$? Unfortunately, the answer is no as shown by the Kahn-Kalai-Linial(KKL) theorem:

Theorem 1.1 (KKL)**.** *If $f\colon \{\pm 1\}^n \to \{\pm 1\}$ and balanced, then there exists $i \in [n]$ such that*

$$I_i(f) = \Omega\left(\frac{\log n}{n}\right).$$

2 Monotone Functions

To understand various voting scenarios further, we now focus on the study of monotone functions.

Definition 2.1. *A boolean-valued function $f\colon \{\pm 1\}^n \to \{\pm 1\}$ is called **monotone** if $f(x) \geq f(y)$ whenever $x \geq y$, i.e., $x_i \geq y_i$ for all $i \in [n]$.*

Lemma 2.2. *Let f be a monotone function. For all $i \in [n]$, $I_i(f) = \hat{f}(i)$.*

Proof. Since f is monotone, we have

$$\begin{aligned}
I_i &= \mathop{\mathbb{E}}_x \left[D_i f(x)^2\right] \\
&= \mathop{\mathbb{E}}_x \left[\frac{f(x^{i\to 1}) - f(x^{i\to -1})}{2}\right] \\
&= \mathop{\mathbb{E}}_x \left[f(x)x_i\right] \\
&= \hat{f}(i).
\end{aligned}$$

□

Corollary 2.3. *If f is monotone, then*

$$I(f) = \sum_{1\leq i\leq n} \hat{f}(i).$$

Question 1. *Which monotone function f maximizes $I(f)$?*

To answer this question, we compute

$$\begin{aligned}
I(f) &= \sum_{1\leq i\leq n} \mathbb{E}\left[f(x)x_i\right] \\
&= \mathop{\mathbb{E}}_{x\sim\{\pm 1\}^n}\left[f(x)\left(\sum_{1\leq i\leq n} x_i\right)\right] \\
&\leq \mathop{\mathbb{E}}_{x\sim\{\pm 1\}^n}\left[\mathsf{MAJ}_n\right] \qquad \text{since } f \text{ is boolean-valued}
\end{aligned}$$

Let $w = \#\{x_i : f(x) = x_i\}$, which can be interpreted as the number of votes that agree with the outcome of a 2-candidate election. Then, note that

$$\sum_{1 \le i \le n} x_i f(x) = w - (w - n)\,.$$

Therefore,

$$\mathbb{E}(w) = \frac{n}{2} + \frac{I(f)}{2}.$$

3 Noise Stability and Sensitivity

It turns out it is often useful to study the sensitivity of boolean functions when the inputs have random noise. For example, we might want to understand how the outcome of an election changes when the votes are recorded incorrectly with some probability.

Towards this end, we now consider the following definitions. Given a fixed $x \in \{\pm 1\}^n$, we sample a $y \sim N_\rho(x)$, where $\rho \in [0, 1]$ is called the noise parameter. Here, $y_i = \epsilon_i x_i$, where

$$\mathbb{P}(\epsilon_i = 1) = \frac{1}{2} + \frac{\rho}{2}, \quad \mathbb{P}(\epsilon_i = -1) = \frac{1}{2} - \frac{\rho}{2}.$$

Another way of thinking about noise is to note that

$$\mathbb{P}(y_i = x_i) = \rho, \quad \mathbb{P}(y_i = -x_i) = 1 - \rho.$$

Definition 3.1. *Given $f : \{\pm 1\}^n \to \mathbb{R}$, we define the noise operator T_ρ with parameter ρ as*

$$T_\rho f(x) = \mathop{\mathbb{E}}_{y \sim N_\rho(x)} [f(y)]\,.$$

Definition 3.2. *We define the noise stability of $f : \{\pm 1\}^n \to \mathbb{R}$ as*

$$\mathsf{NS}(f) = \mathop{\mathbb{E}}_{x \sim \{\pm 1\}^n\, y \sim N_\rho(x)} [f(x) f(y)]\,.$$

If $f \in \{\pm 1\}$, then

$$\mathsf{NS}_\rho(f) = 2 \mathop{\mathbb{P}}_{x \sim \{\pm 1\}^n,\, y \sim N_\rho(x)} [f(x) = f(y)] - 1.$$

Now, let us compute the Fourier spectrum of $T_\rho f$:

$$\begin{aligned} T_\rho f(x) \mathop{\mathbb{E}}_{N_\rho(x)} [f(y)] &= \sum_{S \subseteq [n]} \hat{f}(S) \mathop{\mathbb{E}}_{y \sim N_\rho(x)} [\chi_S(y)] \\ &= \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} \mathbb{E}\,[y_i] \\ &= \sum_{S \subseteq [n]} \hat{f}(S) \rho^{|S|} \chi_S(x) \end{aligned}$$

Lemma 3.3. *By Plancherel's identity, we have*

$$\mathsf{NS}_\rho(f) = \sum_{S \subseteq [n]} \rho^{|S|} \hat{f}(S)^2.$$