CS 6817: Special Topics in Complexity Theory	**Spring 2025**

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1 Review

Recall that for $f : \{-1,1\}^n \to \{-1,1\}$ and $i \in [n]$, the *influence* of coordinate i on f is defined as

$$I_i(f) = \Pr_{x \sim \{-1,1\}^n} \left[f(x) \neq f\left(x^{\oplus i}\right) \right] \tag{1}$$

where $x^{\oplus i}$ indicates the vector $(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$. The *total influence* of f is defined as

$$I(f) = \sum_{i=1}^{n} I_i(f).$$

Recall the (n-dimensional) *Hamming cube* H_n, defined to be the graph with vertex set $\{-1,1\}^n$ and edge set

$$E = \{(x,y) : \Delta(x,y) = 1\}.$$

For $b \in \{-1,1\}$, define

$$A_b = \{x \in \{-1,1\}^n : f(x) = b\}.$$

The *cut* between A_1 and A_{-1} is defined to be the set

$$\mathrm{Cut}(A_1, A_{-1}) = \{(x,y) \in E : x \in A_1 \text{ and } y \in A_{-1}\}.$$

It was previously shown that

$$I(f) = n\frac{|\mathrm{Cut}(A_1, A_{-1})|}{|E|} = \frac{|\mathrm{Cut}(A_1, A_{-1})|}{2^{n-1}}.$$

Example 1.1. *We determine the total influence of the* `AND` *function. Recall that*

$$\mathtt{AND}(x) = \begin{cases} -1 & \text{if } x_i = -1 \text{ for all } i \in [n] \\ 1 & \text{otherwise.} \end{cases}$$

So $\mathrm{Cut}(A_1, A_{-1})$ *consists of the edges of* H_n *incident to the vertex corresponding to the vector with each entry equal to* -1*. There are* n *such edges, so*

$$I(\mathtt{AND}) = \frac{n}{2^{n-1}}.$$

Recall that for $i \in [n]$, the *ith (discrete) derivative operator* D_i maps $f : \{-1,1\}^n \to \mathbb{R}$ to the function $\mathrm{D}_i f : \{-1,1\}^n \to \mathbb{R}$ defined by

$$\mathrm{D}_i f(x) = \frac{f\left(x^{i \to 1}\right) - f\left(x^{i \to -1}\right)}{2}$$

where $x^{i \to b}$ indicates the vector $(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$.

2 Analytic expressions for influence

Definition 2.1. *For $f : \{-1,1\}^n \to \mathbb{R}$ and $i \in [n]$, the influence of coordinate i on f is defined as*

$$I_i(f) = \mathop{\mathbb{E}}_{x \sim \{-1,1\}^n} \left[\mathrm{D}_i f(x)^2 \right] = \| \mathrm{D}_i f \|_2^2.$$

It was previously shown that Definition 2.1 generalizes Equation (1) for $f : \{-1,1\}^n \to \{-1,1\}$.

Proposition 2.2. *For $f : \{-1,1\}^n \to \mathbb{R}$ and $i \in [n]$,*

$$\mathrm{D}_i f(x) = \sum_{\substack{S \subseteq [n] \\ S \ni i}} \hat{f}(S) x^{S - \{i\}}.$$

Proof. For $i \in [n]$ and $S \subseteq [n]$ we have

$$x^S = \prod_{j \in S} x_j = \begin{cases} x_i x^{S-\{i\}} & \text{if } i \in S \\ x^S & \text{if } i \notin S \end{cases}.$$

Below, it is assumed that we are summing over $S \subseteq [n]$. We have

$$\begin{aligned} f\left(x^{i \to 1}\right) &= \sum_{S \ni i} \hat{f}(S) \left(x^{i \to 1}\right)^S + \sum_{S \not\ni i} \hat{f}(S) \left(x^{i \to 1}\right)^S \\ &= \sum_{S \ni i} \hat{f}(S) \left(x^{i \to 1}\right)^{S - \{i\}} + \sum_{S \not\ni i} \hat{f}(S) \left(x^{i \to 1}\right)^S \\ &= \sum_{S \ni i} \hat{f}(S) x^{S - \{i\}} + \sum_{S \not\ni i} \hat{f}(S) x^S \end{aligned}$$

and

$$\begin{aligned} f\left(x^{i \to -1}\right) &= \sum_{S \ni i} \hat{f}(S) \left(x^{i \to -1}\right)^S + \sum_{S \not\ni i} \hat{f}(S) \left(x^{i \to -1}\right)^S \\ &= -\sum_{S \ni i} \hat{f}(S) \left(x^{i \to -1}\right)^{S - \{i\}} + \sum_{S \not\ni i} \hat{f}(S) \left(x^{i \to -1}\right)^S \\ &= -\sum_{S \ni i} \hat{f}(S) x^{S - \{i\}} + \sum_{S \not\ni i} \hat{f}(S) x^S \end{aligned}$$

so

$$\begin{aligned} \mathrm{D}_i f(x) &= \frac{f\left(x^{i \to 1}\right) - f\left(x^{i \to -1}\right)}{2} \\ &= \frac{1}{2} \left(\sum_{S \ni i} \hat{f}(S) x^{S - \{i\}} + \sum_{S \not\ni i} \hat{f}(S) x^S + \sum_{S \ni i} \hat{f}(S) x^{S - \{i\}} - \sum_{S \not\ni i} \hat{f}(S) x^S \right) \\ &= \frac{1}{2} \left(2 \sum_{S \ni i} \hat{f}(S) x^{S - \{i\}} \right) \\ &= \sum_{S \ni i} \hat{f}(S) x^{S - \{i\}}. \end{aligned}$$

□

Proposition 2.3. *For $f : \{-1,1\}^n \to \mathbb{R}$ and $i \in [n]$,*

$$I_i(f) = \sum_{\substack{S\subseteq[n]\\ S\ni i}} \hat{f}(S)^2.$$

Proof. We have

$$\begin{aligned}
I_i(f) &= \|\mathrm{D}_i f\|_2^2 \\
&= \langle \mathrm{D}_i f, \mathrm{D}_i f\rangle \\
&= \left\langle \sum_{\substack{S\subseteq[n]\\ S\ni i}} \hat{f}(S) x^{S-\{i\}}, \sum_{\substack{T\subseteq[n]\\ T\ni i}} \hat{f}(T) x^{T-\{i\}} \right\rangle \\
&= \sum_{\substack{S\subseteq[n]\\ S\ni i}} \hat{f}(S) \sum_{\substack{T\subseteq[n]\\ T\ni i}} \hat{f}(T) \langle x^{S-\{i\}}, x^{T-\{i\}}\rangle \\
&= \sum_{\substack{S\subseteq[n]\\ S\ni i}} \hat{f}(S) \sum_{\substack{T\subseteq[n]\\ T\ni i}} \hat{f}(T) \delta_{S,T} \\
&= \sum_{\substack{S\subseteq[n]\\ S\ni i}} \hat{f}(S)\hat{f}(S) \\
&= \sum_{\substack{S\subseteq[n]\\ S\ni i}} \hat{f}(S)^2
\end{aligned}$$

with the third line by Proposition (2.2) and $\langle x^{S-\{i\}}, x^{T-\{i\}}\rangle = \delta_{S,T}$ by the orthonormality of the characters together with the fact that we are considering S and T with the same element i removed. □

Corollary 2.4. *For $f : \{-1,1\}^n \to \mathbb{R}$,*

$$I(f) = \sum_{S\subseteq[n]} |S| \hat{f}(S)^2.$$

Proof. We have

$$I(f) = \sum_{i=1}^{n} I_i(f) = \sum_{i=1}^{n} \sum_{\substack{S\subseteq[n]\\ S\ni i}} \hat{f}(S)^2 = \sum_{S\subseteq[n]} |S|\hat{f}(S)^2$$

with the rightmost equality because each S contains $|S|$ indices and therefore appears $|S|$ times in the double sum. □

3 Poincaré inequality

Theorem 3.1. *For $f : \{-1,1\}^n \to \mathbb{R}$,* $\mathrm{Var}(f) \leq I(f)$.

Proof. Recall that

$$\mathop{\mathbb{E}}_{x\sim\{-1,1\}^n} \left[f^2\right] = \langle f, f\rangle = \sum_{S\subseteq[n]} \hat{f}(S)^2$$

by Parseval's theorem, and

$$\mathop{\mathbb{E}}_{x\sim\{-1,1\}^n}[f] = \mathop{\mathbb{E}}_{x\sim\{-1,1\}^n}[f1] = \mathop{\mathbb{E}}_{x\sim\{-1,1\}^n}[f\chi_\varnothing] = \langle f, \chi_\varnothing\rangle = \hat{f}(\varnothing)$$

where 1 denotes the identity function and χ_S denotes the indicator function of a set $S \subseteq [n]$. So

$$\begin{aligned}
\mathrm{Var}(f) &= \mathop{\mathbb{E}}_{x\sim\{-1,1\}^n}\left[f^2\right] - \mathop{\mathbb{E}}_{x\sim\{-1,1\}^n}[f]^2 \\
&= \sum_{S\subseteq[n]} \hat{f}(S)^2 - \hat{f}(\varnothing)^2 \\
&\le \sum_{S\subseteq[n]} |S|\hat{f}(S)^2 \\
&= I(f)
\end{aligned}$$

with the inequality because $|S| \geq 1$ for $S \neq \varnothing$, and the final equality by Corollary (2.4). □

4 Influential coordinates for balanced functions

Call $f : \{-1,1\}^n \to \{-1,1\}$ *balanced* if $\mathbb{E}[f] = 0$ and relax this to allow $\mathrm{Var}(f) = \Omega(1)$. In this case, the Poincaré inequality implies that there exists some $i \in [n]$ such that $I_i(f) = \Omega(1/n)$. This motivates the following question.

Question 1. *Does there exist a balanced function f with $\max_i\{I_i(f)\} = O(1/n)$?*

It was previously shown that

$$\max_i\{I_i(\texttt{MAJORITY})\} = \frac{1}{\sqrt{n}}$$

and

$$\max_i\{I_i(\texttt{PARITY})\} = 1.$$

Definition 4.1. *Given $\ell, b \in \mathbb{Z}_{>0}$, the function* `TRIBES`: $\{-1,1\}^{\ell b} \to \{-1,1\}$ *is an* `OR` *of* `AND`*s on $n = \ell b$ variables $\{x_{ij}\}_{1\le i\le \ell, 1\le j\le b}$ defined by*

$$\texttt{TRIBES}(x_{11},\ldots,x_{1b},\ldots,x_{\ell 1},\ldots,x_{\ell b}) = (x_{11}\wedge\cdots\wedge x_{1b}) \vee \cdots \vee (x_{\ell 1}\wedge\cdots\wedge x_{\ell b}).$$

Example 4.2. *We determine the maximal influence of the* `TRIBES` *function. The function is symmetric, so it suffices to find the influence of the first variable x_{11}. This variable is pivotal exactly when $x_{12} = \cdots = x_{1b} = -1$ and the second through ℓth* `AND` *gates equal 1*[1]*, so*

$$I_{11}(\texttt{TRIBES}) = \Pr_{x\sim\{-1,1\}^n}\left[f(x) \neq f\left(x^{\oplus 11}\right)\right] = \frac{1}{2^{b-1}}\left(1 - \frac{1}{2^b}\right)^{\ell-1}.$$

The `TRIBES` *function is roughly balanced for $b \approx \log n - \log\log n$. For such b, one can show that $I_{11} = O(\log n/n)$. So $\max_i\{I_i(\texttt{TRIBES})\} = O(\log n/n)$.*

The KKL theorem, to be stated and proved later, shows that the `TRIBES` example is tight up to a constant factor.

[1] Informally, x_{11} is pivotal when each $x_{12},\ldots,x_{1b}$ is "on" and the second through ℓth `AND` gates are "off". "On" and "off" for $x_{ij} \in \{-1,1\}$ correspond to -1 and 1, respectively.