CS 6817: Special Topics in Complexity Theory **Spring 2025**

Lecture 15: March 20, 2025

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1 Hypercontractivity

Recall the following definitions. For any $x \in \{-1,1\}^n, \rho \in [0,1]$, we have the noisy distribution $N_\rho(x)$ on $-1, 1^n$ samples as follows:

$$y \sim N_\rho(x)$$
$$y_i = \begin{cases} x_i & \text{with probability } \rho \\ \text{random} & \text{with probability } 1-\rho \end{cases}$$

where each y_i is sampled independently.

From this, we define the noise operator $T_\rho f(x)$.

$$\begin{aligned} T_\rho f(x) &= \mathbb{E}_{y\sim N_\rho(x)}[f(y)] \\ &= \sum_{S\subseteq[n]} \rho^{|S|}\hat{f}(S)^2 \end{aligned}$$

Theorem 1.1 ((2, 4)-Hypercontractivity Theorem)**.** *For* $f : \{-1,1\}^n \to \mathbb{R}, \rho = \frac{1}{\sqrt{3}}$, $||T_\rho f||_4 \leq ||f||_2$.

Proof. We will perform induction on n. For the base case $n = 0$, the theorem holds trivially. Next, we perform the inductive step. Assume that the theorem holds for $n-1$. We can write $f : \{-1,1\}^n \to \mathbb{R}$ as the sum of Fourier coefficients without x_n and those with.

$$\begin{aligned} f(x_1,\ldots,x_n) &= \sum_{S\subseteq[n-1]} \hat{f}(S)\chi_S(x) + \sum_{S\subseteq[n-1]} \hat{f}(S\cup\{n\})\chi_S(x)x_n \\ &= f_1(x) + x_n f_2(x) \end{aligned}$$

Here, we take f_1, f_2 to be functions on the first $n-1$ bits. Next, we examine the noise operator on f.

$$\begin{aligned} T_\rho f(x) &= T_\rho(f_1(x) + x_n f_2(x)) \\ &= T_\rho f_1(x) + T_\rho(x_n f_2(x)) \\ &= T_\rho f_1(x) + \rho x_n T_\rho f_2(x) \end{aligned}$$

Note that because T_ρ is just taking the expectation, it is linear. Also, because each bit y_i in the noise operator in sampled independently of the other bits, we can separate x_n from $f_2(x)$ in the noise operator. Now we examine the 4 norm.

$$\begin{aligned} ||T_\rho f||_4^4 &= \mathbb{E}_x[T_\rho f(x)^4] \\ &= \mathbb{E}_x[(T_\rho f_1(x) + \rho x_n T_\rho f_2(x))^4] \\ &= \mathbb{E}_x[T_\rho f_1(x)^4] + 4\mathbb{E}_x[T_\rho f_1(x)^3 \rho x_n T_\rho f_2(x)] + 6\mathbb{E}_x[T_\rho f_1(x)^2(\rho x_n T_\rho f_2(x))^2] \\ &\quad + 4\mathbb{E}_x[T_\rho f_1(x)(\rho x_n T_\rho f_2(x))^3] + \mathbb{E}_x[(\rho x_n T_\rho f_2(x))^4] \end{aligned}$$

Next, observe that we can use the independence of x_n from both $T_\rho f_1, T_\rho f_2$.

$$\mathbb{E}_x[T_\rho f_1(x)^3 \rho x_n T_\rho f_2(x)] = \mathbb{E}_x[T_\rho f_1(x)^3 T_\rho f_2(x)]\mathbb{E}_x[\rho x_n] = 0$$

Note that at the end, since x_n is uniformly randomly sampled from $\{-1, 1\}$, its expectation will be 0. The same logic holds for $\mathbb{E}_x[T_\rho f_1(x)(\rho x_n T_\rho f_2(x))^3] = 0$.

With this, we can further simplify the 4 norm of the noise operator.

$$\begin{aligned}
||T_\rho f||_4^4 &= \mathbb{E}_x[T_\rho f_1(x)^4] + 6\mathbb{E}_x[T_\rho f_1(x)^2(\rho x_n T_\rho f_2(x))^2] + \mathbb{E}_x[(\rho x_n T_\rho f_2(x))^4] \\
&= \mathbb{E}_x[T_\rho f_1(x)^4] + 6\rho^2\mathbb{E}_x[T_\rho f_1(x)^2 T_\rho f_2(x)^2]\mathbb{E}_x[x_n^2] + \rho^4\mathbb{E}_x[T_\rho f_2(x)^4]\mathbb{E}_x[x_n^4] \\
&= \mathbb{E}_x[T_\rho f_1(x)^4] + 2\mathbb{E}_x[T_\rho f_1(x)^2 T_\rho f_2(x)^2] + \rho^4\mathbb{E}_x[T_\rho f_2(x)^4] \\
&\leq \mathbb{E}_x[T_\rho f_1(x)^4] + 2\mathbb{E}_x[T_\rho f_1(x)^2 T_\rho f_2(x)^2] + \mathbb{E}_x[T_\rho f_2(x)^4]
\end{aligned}$$

Note that we use the fact that $x_n \in \{-1, 1\}$ to conclude that $x_n^2 = x_n^4 = 1$ always. Also, $\rho = \frac{1}{\sqrt{3}}$ was used for the last inequality.

Next, we use the Cauchy-Schwartz inequality to simplify the middle term.

$$\mathbb{E}_x[T_\rho f_1(x)^2 T_\rho f_2(x)^2] \leq \left(\mathbb{E}_x[T_\rho f_1(x)^4]\mathbb{E}_x[T_\rho f_2(x)^4]\right)^{\frac{1}{2}}$$

Using this, we can now apply the inductive hypothesis on $T_\rho f_1, T_\rho f_2$.

$$\begin{aligned}
||T_\rho f||_4^4 &\leq \mathbb{E}_x[T_\rho f_1(x)^4] + \left(\mathbb{E}_x[T_\rho f_1(x)^4]\mathbb{E}_x[T_\rho f_2(x)^4]\right)^{\frac{1}{2}} + \mathbb{E}_x[T_\rho f_2(x)^4] \\
&= ||T_\rho f_1||_4^4 + 2||T_\rho f_1||_4^2||T_\rho f_2||_4^2 + ||T_\rho f_2||_4^4 \\
&\leq ||f_1||_2^4 + 2||f_1||_2^2||f_2||_2^2 + ||f_2||_2^4 \\
&= \left(||f_1||_2^2 + ||f_2||_2^2\right)^2 \\
&= ||f||_2^4
\end{aligned}$$

Note that at the very end, we use the fact that f_1, f_2 have disjoint Fourier coefficients by construction. Thus, adding them together will exactly give the full set of Fourier coefficients of f.

Thus, by induction, we have the $(2, 4)$-Hypercontractivity theorem. □

Theorem 1.2 ($(\frac{4}{3}, 2)$-Hypercontractivity Theorem)**.** *For $f : \{-1, 1\}^n \to \mathbb{R}, \rho = \frac{1}{\sqrt{3}}$, $||T_\rho f||_2 \leq ||f||_{\frac{4}{3}}$.*

Proof. We have,

$$\begin{aligned}
||T_\rho f||_2^2 &= \langle T_\rho f, T_\rho f\rangle \\
&= \sum_{S \subseteq [n]} \rho^{2|S|}\hat{f}(S)^2 \\
&= \langle f, T_{\rho^2} f\rangle
\end{aligned}$$

Next, we use Holder's Inequality with $p = \frac{4}{3}, q = 4$.

$$\begin{aligned}||T_\rho f||_2^2 &= \langle f, T_{\rho^2} f\rangle \\ &\leq ||f||_{\frac{4}{3}} ||T_{\rho^2} f||_4 \\ &= ||f||_{\frac{4}{3}} ||T_\rho(T_\rho f)||_4 \\ &\leq ||f||_{\frac{4}{3}} ||T_\rho f||_2 \\ \implies ||T_\rho f||_2 &\leq ||f||_{\frac{4}{3}}\end{aligned}$$

Note that we use the $(2, 4)$ Hypercontractivity theorem for $||T_\rho(T_\rho f)||_4 \leq ||T_\rho f||_2$. □

2 Noisy Hypercube

Definition 2.1. *The noisy hypercube is defined as the following weighted graph $G = (V, E, w)$ where $w : E \to \mathbb{R}$ is the weight of an edge.*

$$V = \{-1, 1\}^n$$
$$w(e = (u, v)) = \mathbb{P}_{y\sim N_\rho(u)}(y = v)$$

Note that with this definition, the set of edges is complete and also includes self-edges.

Observation 2.2. *For any $u \in V$, $\sum_{v\in V} w(u, v) = 1$. Further, for any $u, v \in V$, $w(u, v) = w(v, u)$.*

Definition 2.3. *$G = (V, E, w)$ is a (ϵ, δ) small set expander if for any $S \subseteq [n], |S| \leq \delta|V|$,*

$$\phi_G(S) = \mathbb{P}_{(x\sim S, y\sim w)}(y \notin S) \geq 1 - \epsilon$$

Proposition 2.4. *The noisy hypercube, with $\rho = 1/\sqrt{3}$, is a small set expander.*

Proof. Let $\rho = \frac{1}{\sqrt{3}}, S \subseteq \{-1, 1\}^n, |S| = \delta 2^n$. Define $f : \{-1, 1\}^n \to \mathbb{R}$ as the indicator on S, $f = 1_S$.

Observe that the noise stability of f is related to the small set expander definition.

$$\begin{aligned}Stab_{\rho^2}(f) &= \mathbb{E}_{x\sim\{-1,1\}^n, y\sim N_{\rho^2}(x)}[f(x)f(y)] \\ &= \mathbb{P}(x \in S \wedge y \in S) \\ &= \mathbb{P}(x \in S)\mathbb{P}(y \in S|x \in S)\end{aligned}$$

The conditional probability $\mathbb{P}(y \in S|x \in S)$ is exactly the complement of the small set expander probability $\phi_G(S)$. Thus, we aim to get a bound on the noise stability.

$$\begin{aligned}Stab_{\rho^2}(f) &= \langle f, T_{\rho^2} f\rangle \\ &= \langle T_\rho f, T_\rho f\rangle \\ &= ||T_\rho f||_2^2 \\ &\leq ||f||_{\frac{4}{3}}^2 \\ &= \mathbb{E}[|f(x)|^{\frac{4}{3}}]^{\frac{3}{2}} \\ &= \mathbb{P}(f(x) = 1)^{\frac{3}{2}} \\ &= \left(\frac{|S|}{2^n}\right)^{\frac{3}{2}} \\ &= \delta^{\frac{3}{2}}\end{aligned}$$

For the inequality, we used the $(\frac{4}{3}, 2)$-Hypercontractivity theorem. Note that we also used the fact that f is an indicator and thus only takes values in $\{0, 1\}$ to compute the expectation.

Finally, we can use this to compute the desired probability.

$$\begin{aligned}
\mathbb{P}_{x \sim S, y \sim N_\rho(x)}(y \notin S) &= 1 - \mathbb{P}(y \in S) \\
&= 1 - \frac{Stab_{\rho^2}(f)}{\mathbb{P}(x \in S)} \\
&\geq 1 - \frac{\delta^{\frac{3}{2}}}{\delta} \\
&= 1 - \delta^{\frac{1}{2}}
\end{aligned}$$

Thus, we have shown that the noisy hypercube is a $(\delta^{\frac{1}{2}}, \delta)$ small set expander. □

Remark 2.5. *The bound on* $Stab_{\rho^2}(f)$ *would also work if* f *took values in* $\{0, 1, -1\}$ *rather than just* $\{0, 1\}$*. This will be useful for future hypercontractivity applications.*