CS 6815: Pseudorandomness and Combinatorial Constructions **Fall 2022**

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1 Introduction

Expander graphs are graphs that are both sparse and well connected. By sparse we mean that they have $O(N)$ edges (where N is the number of edges in the graph). There are several different (yet connected) definitions of well connected which we will see thoroughout the lecture.

2 Vertex Expansion

For the duration of the lecture, we will be considering directed D-regular graphs.

Definition 2.1. *For a graph $G = (V, E)$, we define the neighbor set of a vertex $u \in V$ as $N(u) = \{v : (u, v) \in E\}$. We define the neighbor set of a set of vertices $S \subseteq V$ as $N(S) = \bigcup_{u \in S} N(u)$.*

Definition 2.2. *G is a (K, A) vertex expander if for all $S \subseteq V$ such that $|S| \le K$, $|N(S)| \ge A|S|$.*

Notice that the property that $|S| \le K$ is necessary because otherwise $A = 1$ (because then we can have the case where $S = V$).

Intuitively, a G is a vertex expander if when we look at a subset of vertices S, we can always reach more vertices if we take a step from inside the set S.

Remark 2.3. *Edge expansion is very similar, it just requires you to think about the neighbors as the edges you can reach quickly form S rather than the vertices you can reach quickly from S.*

Ideally, we would like $D = O(1)$, $A \approx D - 1$ and $K = \Omega(N)$.

We can show the existence of expanders by the probabilistic method.

However, we will cheat slightly and show the existence of bipartite expanders. A bipartite expander only requires that the expansion property only hold for subsets in the left side of the graph. We will also only require that the graph is left D-regular.

Theorem 2.4. *for all $D = O(1)$, there exists an $\alpha = O(1)$ such that a random left D-regular bipartite digraph is an $(\alpha N, D - 1.1)$ vertex expander with high probability.*

Let us consider the probability that for a fixed set S with $|S| = k$ that a random bipartite graph violates the expansion property. In other words that $|N(S)| \le (D - 1.1)K$.

Notice that for $|N(S)| \le DK - 1.1K$, there must be at least $1.1K$ repetitions in the edges that repeat a node, go to a node which has already been reached by a previous edge.

The probability that a given edge goes to a node that has been covered by a previous edge is bounded above by $\frac{KD}{N}$. Thus, the probability that a given set of edges cover $1.1K$ or more already covered nodes is bounded above by $(\frac{KD}{N})^{1.1K}$. Furthermore, there are $\binom{N}{1.1K}$ choices for the edges which will be repetitions. Thus, for a given S, the probability that there will be $1.1K$ or more repetitions is upper bounded by $\binom{N}{1.1K}(\frac{KD}{N})^{1.1K}$.

Now we will look at the probability that there exists any S with $|S| = k$, which violates the expansion property.

$$\mathbb{P}[\exists S, |S| = K, |N(S)| \leq (D - 1.1)K]$$

By the union bound this is

$$\leq \binom{N}{K}\binom{N}{1.1K}(\frac{KD}{N})^{1.1K}$$

By the approximation for the binomial we can see that the above is less than or equal to

$$\leq (\frac{eN}{K})^{K}(\frac{eKD}{1.1K})^{1.1K}(\frac{KD}{N})^{1.1K}$$

$$= (\frac{e^{2.1}D^{2.2}}{1.1^{1.1}}\frac{K^{0.1}}{N^{0.1}})^{K}$$

By making α very small, we can make K arbitrarily small, thus we can make the following less than or equal to

$$\leq 11^{-K}$$

Finally, we can arrive at the probability that the expansion property is not violated for any set of size less than K as less than or equal to

$$\sum_{i=1}^{\alpha N} 11^{-1} \leq 0.1$$

Thus with high probability (at least 90%) our graph is an expander. Thus, it exists.

3 Spectral Expansion

Definition 3.1. *$\boldsymbol{M}$ is the random walk matrix of a graph G if $\boldsymbol{M}_{i,j} = \frac{\texttt{number of edges from i to j}}{D}$.*

$\mathbf{M}_{i,j}$ can be thought of as the probability you go from to node j if you are at node i and choose to travel along one of the edges with equal probability.

let $\boldsymbol{\pi} = [p_1, p_2, \ldots, p_n]$, where p_i is the probability that you are at node i.

$(\boldsymbol{\pi}\mathbf{M})_i$ is then the probability that you are at node i after taking a random step in the graph having been in node x with probability p_x previously.

let $\mathbf{u}$ be the vector that represents the uniform distribution on vertices, $\mathbf{u} = [\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}]$

Then we can define the expansion of the graph as

$$\lambda(G) = \max_{\mathbf{x} \perp \mathbf{u}} \frac{\|\mathbf{xM}\|}{\|\mathbf{x}\|}$$

Remark 3.2. *Usually $\boldsymbol{xM}^n$ (the distribution after taking n steps in the graph) converges to a stationary distribution.*

$\lambda(G)$ can be thought of as how fast you converge to the uniform distribution for any π. We will now see why.

$\pi = \mathbf{u} + \mathbf{x}$, observe that $\mathbf{u} \perp \mathbf{x}$ since $\|\mathbf{x}\| = 0$

Therefore $\pi\mathbf{M} = (\mathbf{u} + \mathbf{x})\mathbf{M} = \mathbf{u} + \mathbf{xM}$ and we know that $\|\mathbf{xM}^t\| \leq \lambda(G)^t\|\mathbf{x}\|$. So how quickly the distribution converges to the uniform distribution is dependent in $\mathbf{xM}^t$ which is bounded by $\lambda(G)$. So $\lambda(G)$ tells us how quickly the distribution converges to uniform.

Remark 3.3. *If you take an undirected graph and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of M, then $\lambda(G) = \lambda_2$.*

Theorem 3.4. *Spectral expansion implies vertex expansion. More precisly, for $\alpha \in [0,1]$, G is a $(\alpha N, \frac{1}{(1-\alpha)\lambda^2+\alpha})$ vertex expander.*

Notice that if $\lambda < 1$, you have $\frac{1}{(1-\alpha)\lambda^2+\alpha} > 1$, which means we have expansion. Otherwise, we do not.

4 Mixing

Definition 4.1. *A graph G has the mixing property if for 2 sets S and T where $|S| = \alpha N$ and $|T| = \beta N$, $\frac{e(S,T)}{ND} \approx \alpha\beta$. $e(S,T)$ is the number of edges between S and T.*

Notice that in a random graph, you would expect the mixing property to hold.

Theorem 4.2. *Spectral Expansion implies Mixing. More precisely $|\frac{e(S,T)}{ND} - \alpha\beta| \leq \lambda\sqrt{\alpha\beta(1-\alpha)(1-\beta)}$ or the more useful bound $|\frac{e(S,T)}{ND} - \alpha\beta \leq \sqrt{\alpha\beta}$*

Note that if $\lambda \approx 0$, then the density between S and T is very close to $\alpha\beta$.

Remark 4.3. *Vertex expansion with strong parameters implies spectral expansion. In fact, to a certain degree any of the 3 definitions of expansion given in this lecture imply the other 2.*

We will now present the proof that mixing implies spectral expansion
$\mathbb{1}_S = [\mathbb{1}_1, \mathbb{1}_2, \ldots, \mathbb{1}_n]$, where $\mathbb{1}_i = 1$ if and only if $i \in S$.

$$e(S,T) = \mathbb{1}_S^T A \mathbb{1}_T$$

Where A is the adjacency matrix of M. $A = DM$. Therefore

$$e(S,T) = \mathbb{1}_S^T DM \mathbb{1}_T$$

The above is true because the left hand side is equal to the following

$$\sum_{i,j} (\mathbb{1}_S)_i (D\mathbf{M})_{i,j} (\mathbb{1}_T)_j$$

Notice that the expression expression in the sum gives the number of edges between i and j if i is in S and j is in T, and zero otherwise. Thus the sum gives the total number of edges between S and T.

Recall from linear algebra that we can write any vector v as $kv + v^{\perp}$, where $k = \sum v_i$.

Using this fact, we can rewrite out expression as

$$(\alpha N v + (\mathbb{1}_S^{\perp}))^T D\mathbf{M}(\beta N v + (\mathbb{1}_T^{\perp}))^T$$

Expanding and combing terms we get

$$\alpha\beta N^2 D v^T \mathbf{M} v \frac{1}{N} + ((\mathbb{1}_S)^{\perp})^T D\mathbf{M}(\mathbb{1}_T)^{\perp}$$

$$= \alpha\beta N^2 D \frac{1}{N} + ((\mathbb{1}_S)^{\perp})^T D\mathbf{M}(\mathbb{1}_T)^{\perp}$$

The first term now simplifies to $\alpha\beta ND$. All that remains is to bound the error term $((\mathbb{1}_S)^\perp)^T D\mathbf{M}(\mathbb{1}_T)^\perp$. By Cauchy-Swartz

$$((\mathbb{1}_S)^\perp)^T D\mathbf{M}(\mathbb{1}_T)^\perp \leq \|(\mathbb{1}_S)^\perp\| \|D\mathbf{M}\mathbb{1}_T^\perp\|$$

$$\leq \lambda D \|\mathbb{1}_S^\perp\| \|\mathbb{1}_T^\perp\|$$

By Pythagoras, we know $\|\mathbb{1}_S^\perp\| = \sqrt{\alpha(1-\alpha)N}$. Thus we have

$$\leq \lambda D \sqrt{\alpha(1-\alpha)N}\sqrt{\beta(1-\beta)N}$$

$$= \lambda ND\sqrt{\alpha(1-\alpha)\beta(1-\beta)}$$