CS 681 Fall 2001 – Problem Set 4
Due in class Friday, November 9

(1) Consider the following *Broadcast Time* problem. We are given a directed graph $G = (V, E)$, with a designated node $r \in V$ and a designated set of "target nodes" $T \subseteq V - \{r\}$. Each node v has a *switching time* s_v, which is a positive integer.

At time 0, the node r generates a message that it would like every node in T to receive. To accomplish this, we want to find a scheme whereby r tells some of its neighbors (in sequence), who in turn tell some of their neighbors, and so on, until every node in T has received the message. More formally, a *broadcast scheme* is defined as follows. Node r may send a copy of the message to one of its neighbors at time 0; this neighbor will receive the message at time 1. In general, at time $t \geq 0$, any node v that has already received the message may send a copy of the message to one of its neighbors, provided it has not sent a copy of the message in any of the time steps $t - s_v + 1, t - s_v + 2, \ldots, t - 1$. (This reflects the role of the *switching time*; v needs a pause of $s_v - 1$ steps between successive sendings of the message. Note that if $s_v = 1$, then no restriction is imposed by this.)

The *completion time* of the broadcast scheme is the minimum time t by which all nodes in T have received the message. The *Broadcast Time* problem is the following: given the input described above, and a bound b, is there a broadcast scheme with completion time at most b?

Prove that *Broadcast Time* is NP-complete.

Example. Suppose we have a directed graph $G = (V, E)$, with $V = \{r, a, b, c\}$; edges (r, a), (a, b), (r, c); the set $T = \{b, c\}$; and switching time $s_v = 2$ for each $v \in V$. Then a broadcast scheme with minimum completion time would be as follows: r sends the message to a at time 0; a sends the message to b at time 1; r sends the message to c at time 2; and the scheme completes at time 3 when c receives the message. (Note that a can send the message as soon as it receives it at time 1, since this is its first sending of the message; but r cannot send the message at time 1 since $s_r = 2$ and it sent the message at time 0.)

(2) Recall from an earlier Problem Set that given a directed graph G with a root and a cost of either 0 or 1 on each edge, one could determine whether G had an arborescence of cost exactly k.

Here's a generalization of this problem. Again we are given a directed graph G with a root; but now each edge has an arbitrary non-negative integer cost. We are also given a value k. The question is whether G has an arborescence of cost exactly k.

Prove that this more general problem is NP-complete.

(3) A *k-coloring* of an undirected graph $G = (V, E)$ is an assignment of one of the numbers $\{1, 2, \ldots, k\}$ to each node, so that if two nodes are joined by an edge, then they are assigned different numbers. The *chromatic number* of G is the minimum k such that it has a k-coloring. For $k \geq 3$, it is NP-complete to decide whether a given input graph has chromatic number $\leq k$. (You don't have to prove this.)

(a) Show that for every natural number $w \geq 1$, there is a number $k(w)$ so that the following holds. If G is a graph of tree-width at most w, then G has chromatic number at most $k(w)$. (The point is that $k(w)$ depends only on w, not on the number of nodes in G.)

(b) Given an undirected n-node graph $G = (V, E)$ of tree-width at most w, show how to compute the chromatic number of G in time $O(f(w) \cdot p(n))$, where $p(\cdot)$ is a polynomial but $f(\cdot)$ can be an arbitrary function.