CS 681 Fall 2001 – Final Exam

The midterm is due at 2:30 pm on Monday, December 17; it can be handed in to me, Niranjan, or Esha Molette (4119 Upson). You can contact me (either by e-mail or in person) or Niranjan with any questions; you are not allowed to discuss the exam with anyone else.

(1) Suppose we are given a set V of vertices, and two graphs defined on V: a directed graph $G = (V, E)$ and an undirected forest $H = (V, E')$. (I.e. G can be an arbitrary directed graph, but the undirected graph H should have no cycles.)

Here is a decision problem. Given G and H, and two designated vertices $s, t \in V$, is there an s-t path P in G so that the nodes in P form an independent set in H? Show that this problem is NP-complete.

(2) Suppose that a tree network is grown according to the following randomized process. We begin in step 1, with a single isolated node v_1. In any step $k \geq 2$, we introduce a new node v_k and draw an edge from v_k to a node chosen uniformly at random from $v_1, v_2, \ldots, v_{k-1}$. We stop after step n, with a tree on n nodes.

What is the expected number of leaves in the resulting tree?

(3) Suppose we are given a set of points $P = \{p_1, p_2, \ldots, p_n\}$, together with a distance function d on the set P; as usual, d is simply a function on pairs of points in P with the properties that $d(p_i, p_j) = d(p_j, p_i) > 0$ if $i \neq j$, and that $d(p_i, p_i) = 0$ for each i.

We define a *stratified metric* on P to be any distance function τ that can be constructed as follows. We build a rooted tree T with n leaves, and we associate with each node v of T (both leaves and internal nodes) a *height* h_v. These heights must satisfy the properties that $h(v) = 0$ for each leaf v, and if u is the parent of v in T, then $h(u) \geq h(v)$. We place each point in P at a distinct leaf in T. Now, for any pair of points p_i and p_j, their distance $\tau(p_i, p_j)$ is defined as follows. We determine the least common ancestor v in T of the leaves containing p_i and p_j, and define $\tau(p_i, p_j) = h_v$.

We say that a stratified metric τ is *consistent* with our distance function d if for all pairs i, j, we have $\tau(p_i, p_j) \leq d(p_i, p_j)$.

Give a polynomial-time algorithm that takes the distance function d and produces a stratified metric τ with the following properties:

(i) τ is consistent with d, and

(ii) if τ' is any other stratified metric consistent with d, then $\tau'(p_i, p_j) \leq \tau(p_i, p_j)$ for each pair of points p_i and p_j.