

The Kernel Trick, Gram Matrices, and Feature Extraction

CS6787 Lecture 4 — Spring 2026

Today: A “Classic” Example of a Tradeoff

- A pre-deep-learning-revolution concept
- **Q: How do to kernel learning?**
- Many ways to set up the computation
 - are mathematically equivalent,
 - but not computationally equivalent
 - or numerically equivalent

Basic Linear Models

- For two-class classification using model vector \mathbf{w}
output = $\text{sign}(w^T x)$
- What is the compute cost of making a prediction in a **d**-dimensional linear model, given an example \mathbf{x} ?
- Answer: **d multiplies and d adds**
 - To do the dot product.

Optimizing Basic Linear Models

- For classification using model vector \mathbf{w}

$$\text{output} = \text{sign}(\mathbf{w}^T \mathbf{x})$$

- Optimization methods for this task vary; here's logistic regression

$$\text{minimize}_{\mathbf{w}} \frac{1}{n} \sum_{i=1}^n \log (1 + \exp(-\mathbf{w}^T \mathbf{x}_i y_i))$$
$$(y_i \in \{-1, 1\})$$

SGD on Logistic Regression

- Gradient of a training example is

$$\nabla f_i(w) = \frac{-x_i y_i}{1 + \exp(w^T x_i y_i)}$$

- So SGD update step is

$$w_{t+1} = w_t + \alpha_t \frac{-x_i y_i}{1 + \exp(w_t^T x_i y_i)}$$

Diagram annotations in red:

- Red circle around w_t with a red arrow pointing to the addition operator.
- Red circle around α_t with a red arrow pointing to the scaling factor.
- Red circle around the term $x_i y_i$ with a red arrow pointing to the gradient component.
- Red circle around the denominator $1 + \exp(w_t^T x_i y_i)$ with a red arrow pointing to the denominator.
- Red annotations at the bottom: "2d" with a red arrow pointing to the denominator, and "2d" with a red arrow pointing to the term $x_i y_i$.

What is the compute cost of an SGD update?

- For logistic regression on a \mathbf{d} -dimensional model

$$w_{t+1} = w_t + \alpha_t \frac{x_i y_i}{1 + \exp(w_t^T x_i y_i)}$$

- Answer: **2d multiples and 2d adds + O(1) extra ops**

- d multiplies and d adds to do the dot product
- d multiplies and d adds to do the AXPY operation
- O(1) additional ops for computing the exp, divide, etc.

Benefits of Linear Models

- **Fast classification:** just one dot product
- **Fast training/learning:** just a few basic linear algebra operations
- **Drawback: limited expressivity**
 - Can only capture linear classification boundaries → bad for many problems
- How do we let linear models **represent a broader class of decision boundaries**, while **retaining the systems benefits?**

Review: The Kernel Method

- Idea: in a linear model we can think about the **similarity** between two training examples \mathbf{x} and \mathbf{y} as being

$$\mathbf{x}^T \mathbf{y}$$

- This is related to the rate at which a random classifier will separate \mathbf{x} and \mathbf{y}
- Kernel methods replace this dot-product similarity with an arbitrary **Kernel function** that computes the similarity between \mathbf{x} and \mathbf{y}

$$K(x, y) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$$

Kernel Properties

- **What properties do kernels need to have to be useful for learning?**
- Key property: kernel must be **symmetric** $K(x, y) = K(y, x)$

- Key property: kernel must be **positive semi-definite**

$$\forall c_i \in \mathbb{R}, x_i \in \mathcal{X}, \sum_{i=1}^n \sum_{j=1}^n c_i c_j K(x_i, x_j) \geq 0$$

- Can check that the dot product has this property

Facts about Positive Semidefinite Kernels

- Sum of two PSD kernels is a PSD kernel

$K(x, y) = K_1(x, y) + K_2(x, y)$ is a PSD kernel

- Product of two PSD kernels is a PSD kernel

$K(x, y) = K_1(x, y)K_2(x, y)$ is a PSD kernel

- Scaling by any function on both sides is a kernel

$K(x, y) = f(x)K_1(x, y)f(y)$ is a PSD kernel

Other Kernel Properties

- Useful property: kernels are often **non-negative**

$$K(x, y) \geq 0$$

- Useful property: kernels are often **scaled** such that

$$K(x, y) \leq 1, \text{ and } K(x, y) = 1 \Leftrightarrow x = y$$

- These properties capture the idea that the kernel is expressing the similarity between **x** and **y**

Common Kernels

- **Gaussian kernel/RBF kernel**: de-facto kernel in machine learning

$$K(x, y) = \exp(-\gamma \|x - y\|^2)$$

- We can validate that this is a kernel
 - Symmetric?
 - Positive semi-definite?
 - Non-negative?
 - Scaled so that $K(x, x) = 1$?

Common Kernels (continued)

- **Linear kernel:** just the inner product $K(x, y) = x^T y$
- **Polynomial kernel:** $K(x, y) = (1 + x^T y)^p$
- **Laplacian kernel:** $K(x, y) = \exp(-\beta \|x - y\|_1)$

Kernels as a feature mapping

- More generally, any function that can be written in the form

$$K(x, y) = \phi(x)^T \phi(y)$$

(where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^D$ is called a feature map) is a kernel.

- Even works for maps onto infinite dimensional **Hilbert space**
 - And in this case the converse is also true: any kernel has an associated (possibly infinite-dimensional) feature map.

Classifying with Kernels

- Recall the SGD update is

$$w_{t+1} = w_t + \alpha_t \frac{x_i y_i}{1 + \exp(w_t^T x_i y_i)}$$

- **Resulting weight vectors will always be in the span of the examples.**

- So, our prediction will be:

$$w = \sum_{i=1}^n u_i x_i \Rightarrow h_w(x) = \text{sign}(w^T x) = \text{sign} \left(\sum_{i=1}^n u_i \underbrace{x_i^T x}_{\text{red}} \right)$$

Classifying with Kernels

- An equivalent way of writing a linear model on a training set is

$$h_w(x) = \text{sign} \left(\sum_{i=1}^n u_i x_i^T x \right)$$

- We can kernel-ize this by replacing the dot products with kernel evaluations

$$h_u(x) = \text{sign} \left(\sum_{i=1}^n u_i K(x_i, x) \right)$$

Learning with Kernels

- An equivalent way of writing linear-model logistic regression is

$$\text{minimize}_u \frac{1}{n} \sum_{i=1}^n \log \left(\frac{1}{1 + \exp \left(- \left(\sum_{j=1}^n u_j x_j \right)^T x_i y_i \right)} \right)$$

- We can kernel-ize this by replacing the dot products with kernel evaluations

$$\text{minimize}_u \frac{1}{n} \sum_{i=1}^n \log \left(\frac{1}{1 + \exp \left(- \sum_{j=1}^n u_j y_j K(x_j, x_i) \right)} \right)$$

The Computational Cost of Kernels

- Recall: benefit of learning with kernels is that **we can express a wider class of classification functions**
- Recall: another benefit is **linear classifier learning problems are “easy”** to solve because they are convex, and gradients easy to compute
- **Major cost of learning naively with Kernels:** have to evaluate $\mathbf{K}(\mathbf{x}, \mathbf{y})$
 - For SGD, need to do this effectively \mathbf{n} times per update
 - Computationally intractable unless \mathbf{K} is very simple

The Gram Matrix

- Address this computational problem by **pre-computing the kernel function** for all pairs of training examples in the dataset.

$$G_{i,j} = K(x_i, x_j)$$

- Transforms the logistic regression learning problem into

$$\text{minimize}_u \frac{1}{n} \sum_{i=1}^n \log (1 + \exp (-y_i e_i^T G u))$$

- This is much easier than re-computing the kernel at each iteration

Problems with the Gram Matrix

- Suppose we have **n** examples in our training set.
- **How much memory** is required to store the Gram matrix **G**?
- **What is the cost** of taking the product **G_i w** to compute a gradient?
- What happens if we have **one hundred million training examples**?

Feature Extraction

- Simple case: let's imagine that \mathbf{X} is a finite set $\{1, 2, \dots, k\}$
- We can define our kernel as a matrix $M \in \mathbb{R}^{k \times k}$

$$M_{i,j} = K(i, j)$$

- Since M is positive semidefinite, it has a square root

$$\sum_{i=1}^k U_{k,i} U_{k,j} = M_{i,j} = K(i, j) \quad U^T U = M$$

Feature Extraction (continued)

- So if we define a **feature mapping** $\phi(i) = Ue_i$ then

$$\underbrace{\phi(i)^T \phi(j)}_{\substack{\longrightarrow \\ \longrightarrow}} = \sum_{i=1}^k U_{k,i} U_{k,j} = M_{i,j} = \underbrace{K(i, j)}_{\substack{\longrightarrow \\ \longrightarrow}}$$

- The kernel is **equivalent to a dot product** in some space
- As we noted above, this is **true for all kernels**, not just finite ones
 - Just with a possibly infinite-dimensional feature map

Classifying with feature maps

- Suppose that we can find a finite-dimensional feature map that satisfies

$$\phi(i)^T \phi(j) = K(i, j)$$

- Then we can simplify our classifier to

$$\begin{aligned} h_u(x) &= \text{sign} \left(\sum_{i=1}^n u_i K(x_i, x) \right) \\ &= \text{sign} \left(\sum_{i=1}^n u_i \phi(x_i)^T \phi(x) \right) = \text{sign} (w^T \phi(x)) \end{aligned}$$

Learning with feature maps

- Similarly we can simplify our learning objective to

$$\text{minimize}_w \frac{1}{n} \sum_{i=1}^n \log \left(1 + \exp \left(- \sum_{j=1}^n w^T \phi(x_i) y_i \right) \right)$$

~~$\sum_{j=1}^n$~~

- Take-away: this is just **transforming the input data, then running a linear classifier in the transformed space!**
- Computationally: **super efficient**
 - As long as we can transform and store the input data in an efficient way

Problems with feature maps

- The dimension of the transformed data may be **much larger than the dimension of the original data.**
- Suppose that the feature map is $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^D$ and there are **n** examples
- **How much memory is needed** to store the transformed features?
- **What is the cost** of taking the product $u^T \phi(x_i)$ to compute a gradient?

Feature maps vs. Gram matrices

- **Interesting systems trade-offs exist here.**
- When number of examples gets very large, **feature maps are better.**
- When transformed feature vectors have high dimensionality, **Gram matrices are better.**

Another Problem with Feature Maps

- Recall: I said there was always a feature map for any kernel such that

$$\phi(i)^T \phi(j) = K(i, j)$$

- But this feature map is **not always finite-dimensional**
 - For example, the Gaussian/RBF kernel has an infinite-dimensional feature map
 - **Many kernels we care about in ML have this property**
- What do we do if ϕ has infinite dimensions?
 - **We can't just compute with it normally!**

Solution: Approximate feature maps

- Find a finite-dimensional feature map so that

$$K(x, y) \approx \phi(x)^T \phi(y)$$

- Typically, we want to find a family of feature maps Φ_t such that

$$\phi_D : \mathbb{R}^d \rightarrow \mathbb{R}^D$$

$$\lim_{D \rightarrow \infty} \phi_D(x)^T \phi_D(y) = K(x, y)$$

Types of approximate feature maps

- **Deterministic feature maps**

- Choose a fixed-a-priori method of approximating the kernel
- Generally not very popular because of the way they scale with dimensions

- **Random feature maps**

- Choose a feature map at random (typically each feature is independent) such that

$$\mathbf{E} [\phi(x)^T \phi(y)] = K(x, y)$$

- Then prove with high probability that over some region of interest

$$|\phi(x)^T \phi(y) - K(x, y)| \leq \epsilon$$

Types of Approximate Features (continued)

- **Orthogonal randomized feature maps**

- Intuition behind this: if we have a feature map where for some i and j

$$e_i^T \phi(x) \approx e_j^T \phi(x)$$

then we can't actually learn much from including both features in the map.

- Strategy: choose the feature map at random, but subject to the constraint that the features be statistically “orthogonal” in some way.

- **Quasi-random feature maps**

- Generate features using a low-discrepancy sequence rather than true randomness

Adaptive Feature Maps

- Everything before this **didn't take the data into account**
- **Adaptive feature maps** look at the actual training set and try to minimize the kernel approximation error using the training set as a guide
 - For example: we can do a random feature map, and then **fine-tune the randomness** to minimize the empirical error over the training set
 - Gaining in popularity
- Also, neural networks can be thought of as adaptive feature maps.

Summary: Many Ways to Learn Linear Models

- **Options for representing features:**

- Learn with an exact feature map
- Learn with a kernel
- Learn with an approximate feature map

- **Other choices:**

- Pre-compute feature map/Gram matrix and store in memory
- Re-compute feature map/kernel value at each iteration

Systems Tradeoffs

- Lots of tradeoffs here
- Do we spend more work up-front constructing a more sophisticated approximation, to save work on learning algorithms?
- Would we rather scale with the data, or scale to more complicated problems?
- Another task for **hyperparameter optimization**

Demo

Is this still used today?

- Yes!
- People still use this trick in low-dimensional applications such as 3D modeling/graphics
- The basis for efficient “linear attention” mechanisms

Dimensionality reduction

Linear models are linear in the dimension

- But what if the dimension \mathbf{d} is very large?
 - Example: if we have a high-dimensional kernel map
- It can be **difficult to run SGD** when the dimension is very high even if the cost is linear
 - This happens for other learning algorithms too

Idea: reduce the dimension

- If high dimension is the problem, can we just reduce **d**?
- This is the problem of **dimensionality reduction**.
- Dimensionality reduction benefits both statistics and systems
 - Statistical side: can **help with generalization** by identifying important subset of features
 - Systems side: lowers compute cost

Techniques for dimensionality reduction

- **Feature selection by hand**
 - Simple method
 - But costly in terms of human effort
- **Principal component analysis (PCA)**
 - Identify the **directions of highest variance** in the dataset
 - Then project onto those directions
 - Many variants: e.g. kernel PCA

More techniques for dimensionality reduction

- **Locality-sensitive hashing** (LSH)
 - Hash input items into buckets so close-by elements map into the same buckets with high probability
 - Many methods of doing this too
- **Johnson-Lindenstrauss transform** (random projection)
 - General method for reducing dimensionality of any dataset
 - Just **choose a random subspace** and project onto that subspace

Johnson-Lindenstrauss lemma

Given a desired error $\epsilon \in (0, 1)$, a set of m points in \mathbb{R}^d , and a reduced dimension D that satisfies $D > \frac{8 \log(m)}{\epsilon^2}$, there exists a linear map T such that

$$(1 - \epsilon) \cdot \|x - y\|^2 \leq \|T(x) - T(y)\|^2 \leq (1 + \epsilon) \cdot \|x - y\|^2$$

for all points x and y in the set.

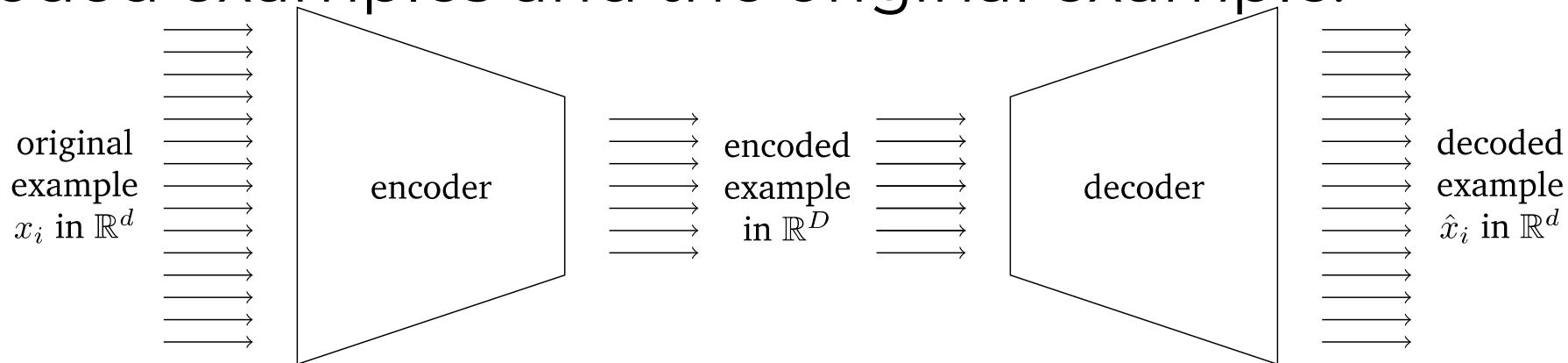
In fact, a randomly chosen linear map T works with high probability!

Consequences of J-L transform

- We only need $O(\log(m) / \epsilon^2)$ dimensions to map a dataset of size m with relative distance accuracy.
 - **No matter what the size of the input dataset was!**
- This is a very useful result for many applications
 - Provides a generic way of reducing the dimension with guarantees
- But there are more **specialized data-dependent ways of doing dimensionality reduction** that can work better.

Autoencoders

- Use **deep learning** to learn two models
 - The **encoder**, which maps an example to a dimension-reduced representation
 - The **decoder**, which maps it back
- Train to minimize the distance between encoded-and-decoded examples and the original example.



Questions

- Upcoming things:
 - **Paper 1a or 1b review due on Monday**
 - Papers 2a/2b in class on Monday
 - Start thinking about the class project
 - It will come faster than you think!