# *Iterative* Linear Quadratic Regulator

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LQR is cute...
But what if my
robot is not linear?


EVERY SINGLE THING ON EARTH IS
EITHER BANANAS
OR NOT BANANAS
made with mematic

LQR is fundamentally a way to *locally* approximate and update value functions

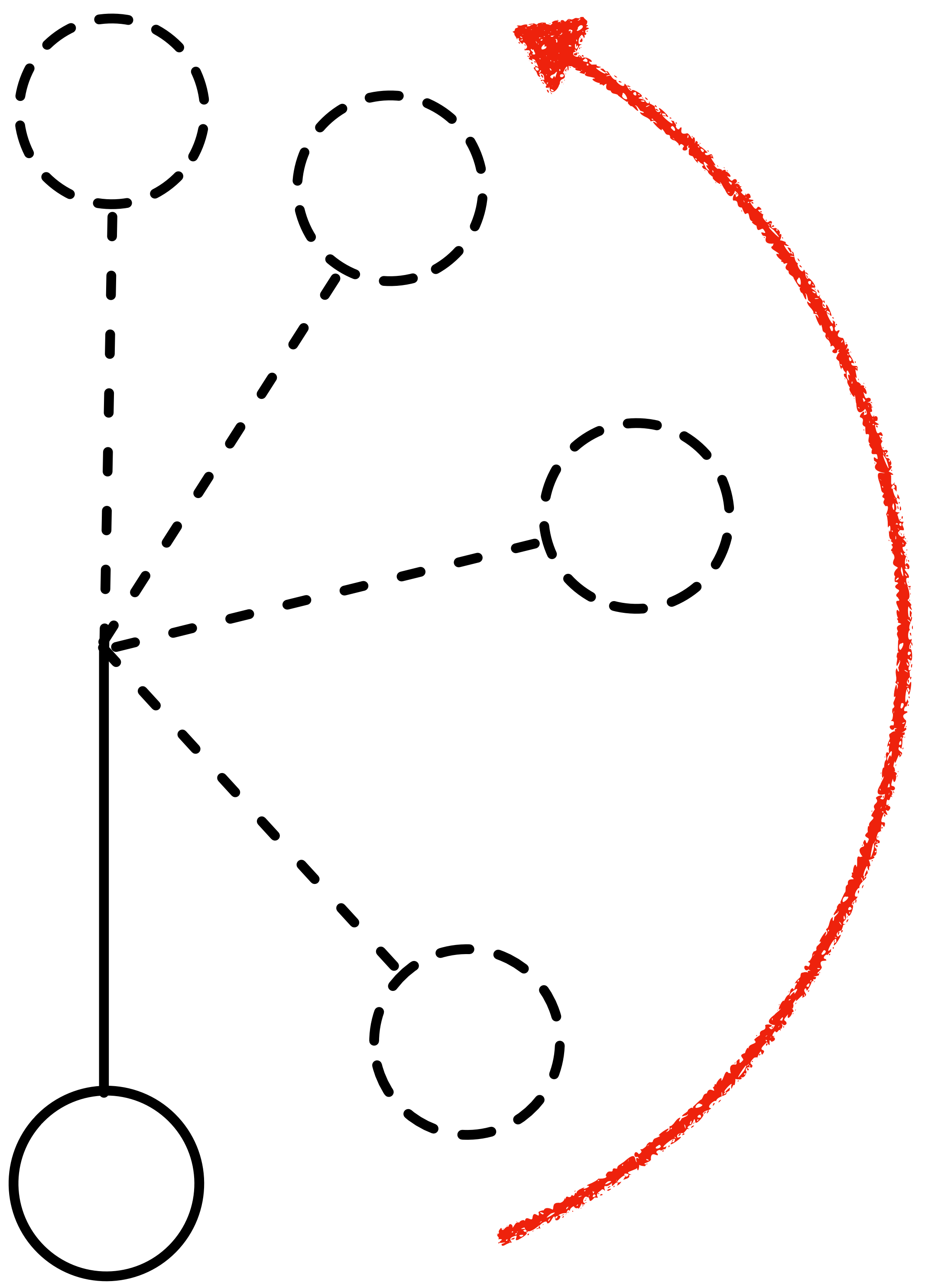
# Activity!

# Think-Pair-Share!

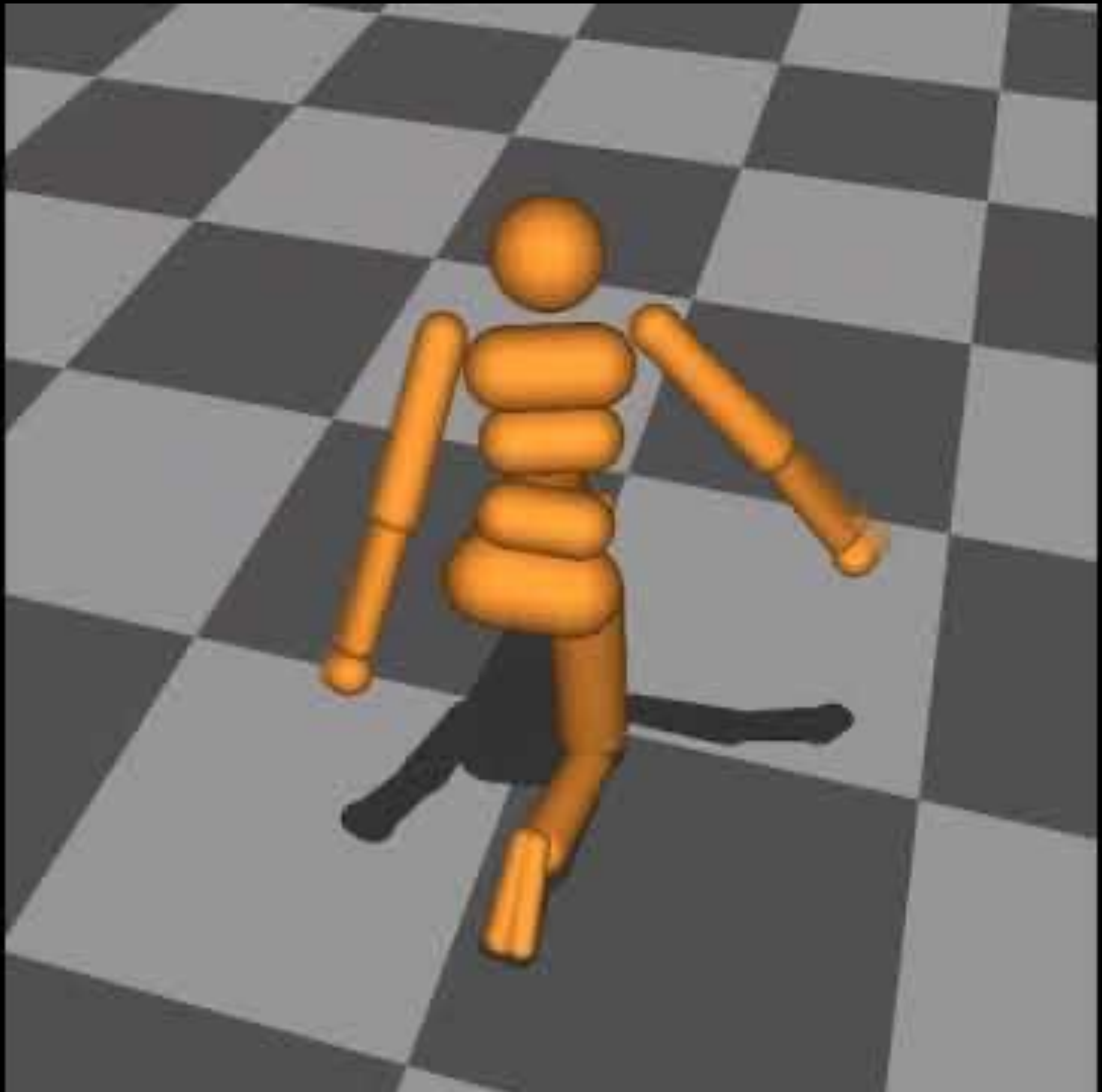
Think (30 sec): How can we use LQR to swing up a pendulum and stabilize it there? What does the optimal solution look like?

Pair: Find a partner

Share (45 sec): Partners exchange ideas

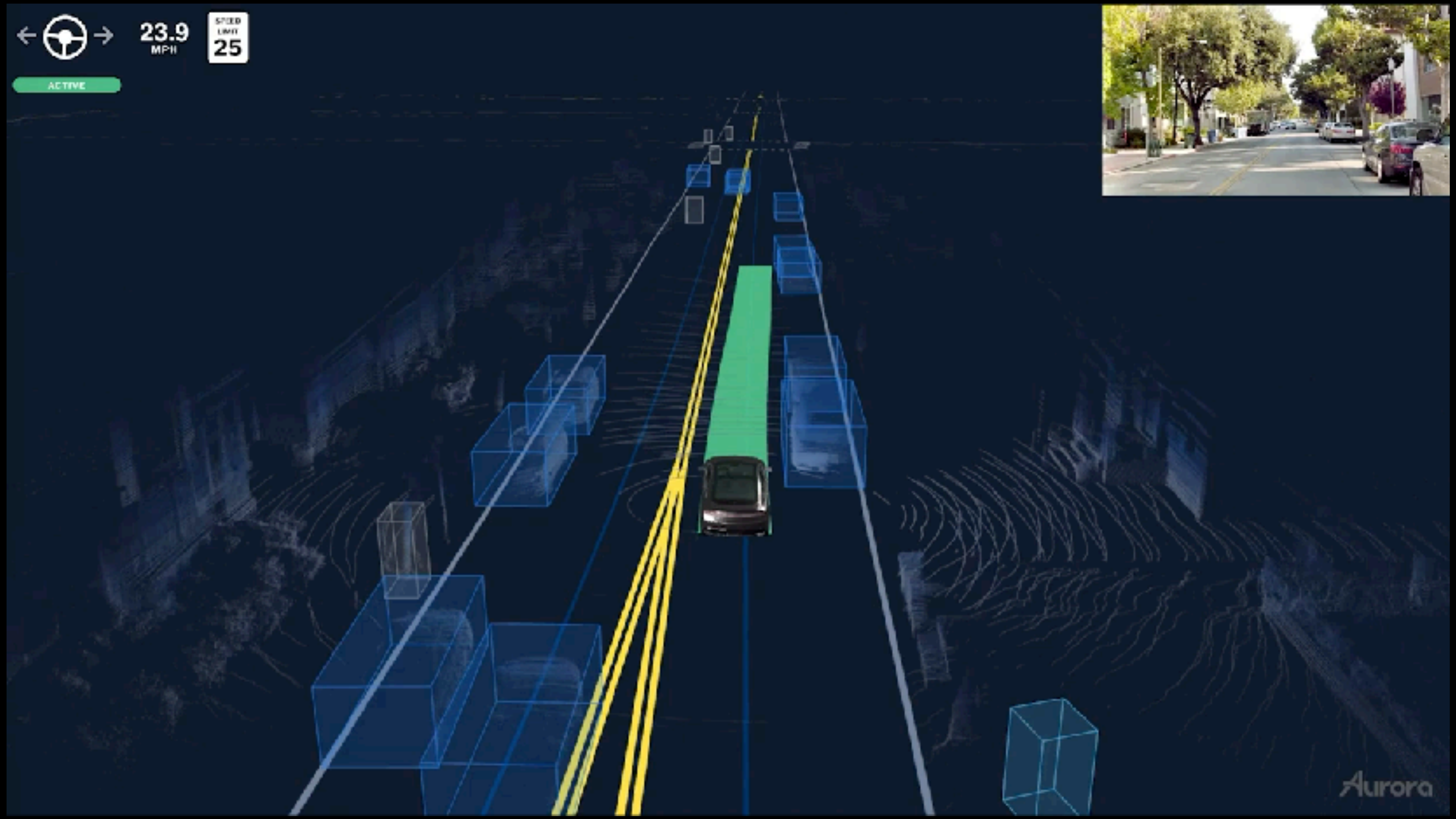


# Iterative LQR (iLQR)



# iLQR in action!


Abbeel et al




Aurora

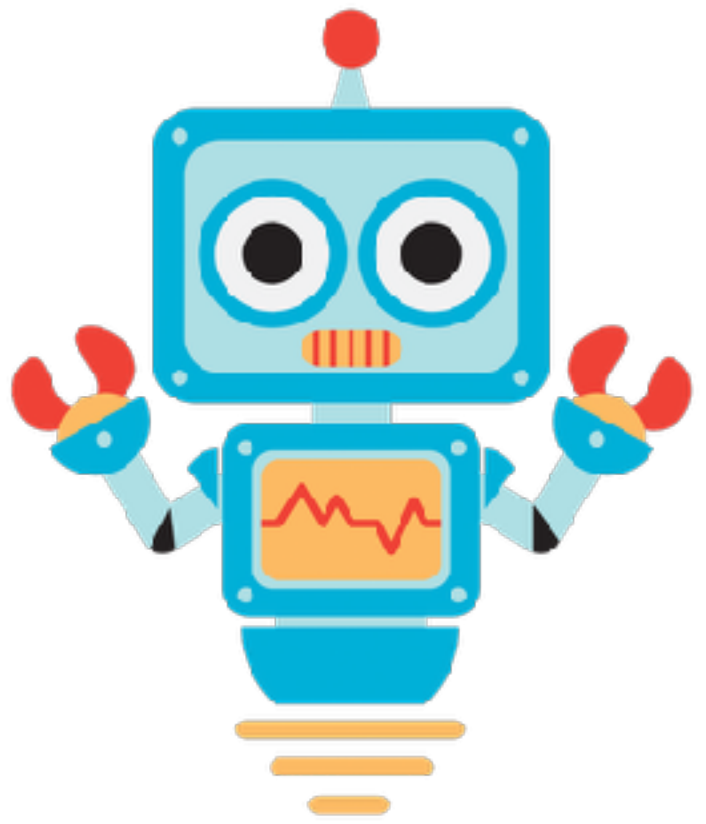
Whole-Arm Manipulation

Target Position: 0.2 m forward

Kurtz and Lin
https://arxiv.org/pdf/2202.13986.pdf

# Iterative LQR (ILQR)

Goal: Solve a *general* continuous time MDP



Nonlinear!

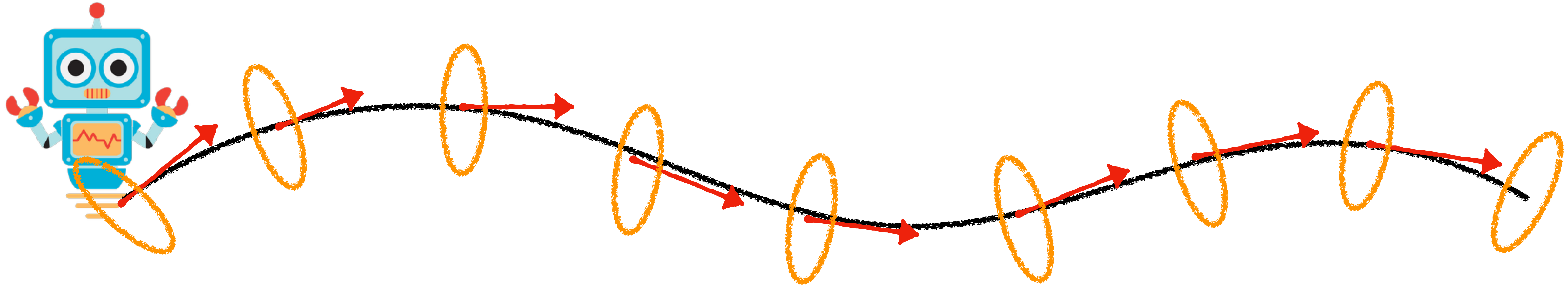
$$\min_{x_{0:T-1}, u_{0:T-1}} \sum_{t=0}^{T-1} c(x_t, u_t)$$

$$x_{t+1} = f(x_t, u_t)$$

Nonlinear!

# Iterative LQR (ILQR) - Spill the beans!

Three simple steps!



Step 1: Forward pass - roll out current guess $u(t)$

Step 2: Linearize dynamics, quadricize cost around roll out

Step 3: Backwards pass - compute LQR gains $K_t$ at each time

# How I learned ILQR ..

Suffer through a barrage of matrix derivations!

(And god forbid you flip a sign...)

*Dynamic Programming (Value-Iteration) Backup*

Assume we have now a control policy of the form of a "feedforward" update term $k_t$ and feedback term $K_T$ that is a linear controller response to "errors" in $z_T$:

$$v_T = K_T z_T + k_T, \tag{2.7.1}$$

Inductively, we assume the next-state value function (i.e. of the future timestep) can be written in the form,

$$J_{T+1} = \frac{1}{2} z_{T+1} V_{T+1} z_{T+1} + G_{T+1} z_{T+1} + W_{T+1}. \tag{2.7.2}$$

Since

$$\begin{aligned} z_{T+1} &= A z_T + B v_T && (2.7.3)\\ &= A z_T + B(K_T z_T + k_T) && (2.7.4)\\ &= (A + BK_T) z_T + B k_T, && (2.7.5) \end{aligned}$$

we can write, $J_{T+1}$ as:

$$\begin{aligned} J_{T+1} &= \frac{1}{2}((A+BK_T)z_T + Bk_T)^T V_{T+1}((A+BK_T)z_T + Bk_T) + G_{T+1}((A+BK_T)z_T + Bk_T) + W_{T+1} && (2.7.6)\\ &= \frac{1}{2} z_T^T (A+BK_T)^T V_{T+1}(A+BK_T) z_T + \frac{1}{2} k_T^T B^T V_{T+1} B k_T + k_T^T B^T V_{T+1}(A+BK_T) z_T && (2.7.7)\\ &\quad + G_{T-1}(A+BK_T) z_T + G_{T-1} B k_T + W_{T-1} && (2.7.8)\\ &= \frac{1}{2} z_T^T (A+BK_T)^T V_{T+1}(A+BK_T) z_T + \left(k_T^T B^T V_{T+1}(A+BK_T) + G_{T+1}(A+BK_T)\right) z_T && (2.7.9)\\ &\quad + G_{T-1} B k_T + \frac{1}{2} k_T^T B^T V_{T-1} B k_T + W_{T-1} && (2.7.10) \end{aligned}$$

Additionally, we can write the cost $c_T(z_T, v_T)$ as:

$$\begin{aligned} c_T &= \frac{1}{2} z_T^T Q z_T + z_T^T P v_T + \frac{1}{2} v_T^T R v_T + g_x^T z_T + g_u^T v_T + c + J_{T+1} && (2.7.11)\\ &= \frac{1}{2} z_T^T Q z_T + z_T^T P(K_T z_T + k_T) + \frac{1}{2}(K_T z_T + k_T)^T R (K_T z_T + k_T) + g_x^T z_T + g_u^T (K_T z_T + k_T) + c && (2.7.12)\\ &= \frac{1}{2} z_T^T Q z_T + z_T^T P K_T z_T + k_T^T P^T z_T + \frac{1}{2} z_T^T K_T^T R K_T z_T + \frac{1}{2} k_T^T R k_T + k_T^T R K_T z_T + g_x^T z_T && (2.7.13)\\ &\quad + g_u^T K_T z_T + g_u^T k_T + c \\ &= \frac{1}{2} z_T^T \left(Q + 2PK_T + K_T^T R K_T\right) z_T + \left(k_T^T P^T + k_T^T R K_T + g_x^T + g_u^T K_T\right) z_T + \frac{1}{2} k_T^T R k_T + g_u^T k_T + c && (2.7.14) \end{aligned}$$

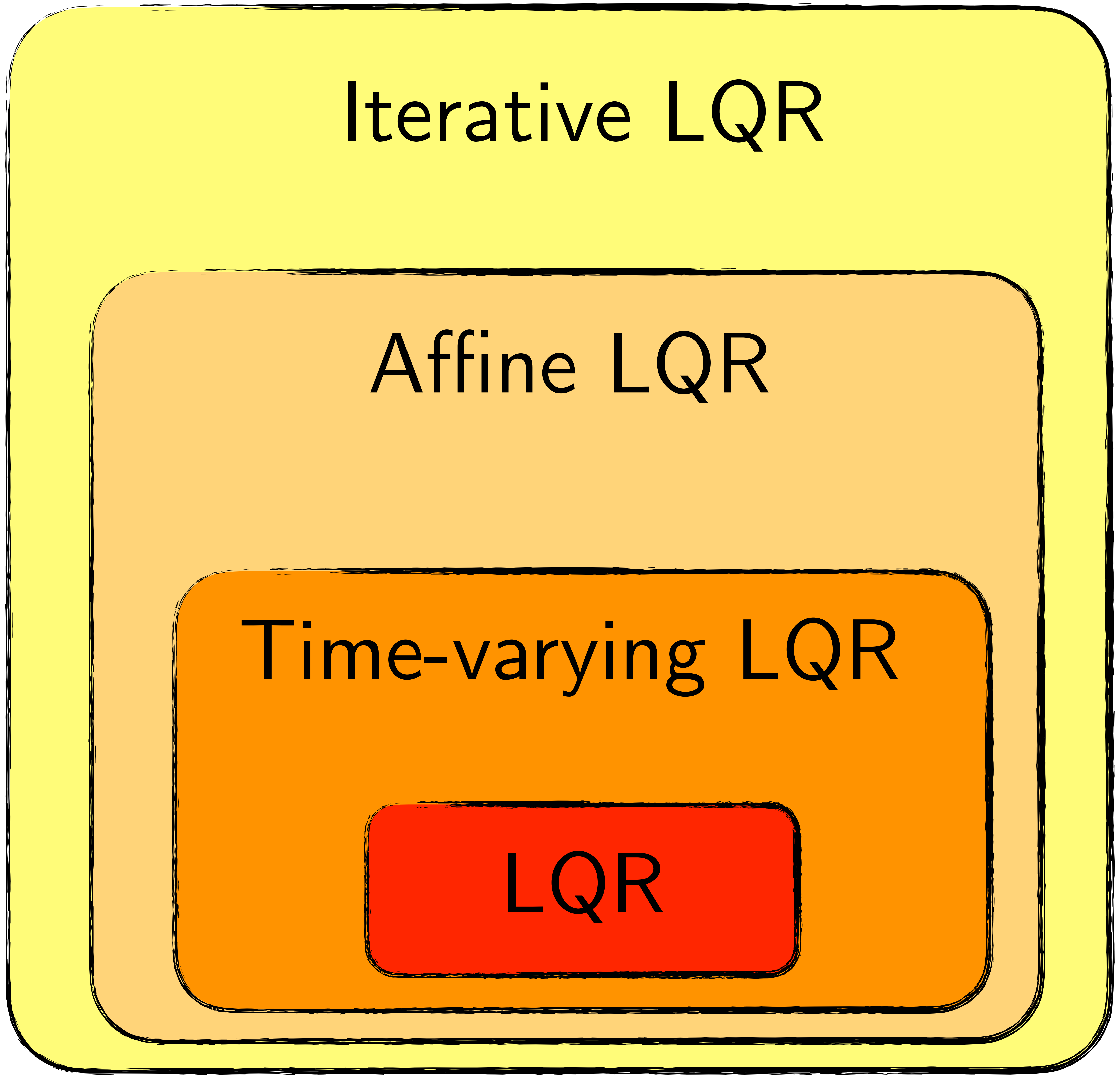
Then, we can write $J_T = c_T(z_T, v_T) + J_{T+1} = \frac{1}{2} z_T^T V_T z_T + G_T z_T + W_T$ by combining like terms from above, where

$$\begin{aligned} V_T &= Q + 2PK_T + K_T^T R K_T + (A+BK_T)^T V_{T+1}(A+BK_T) && (2.7.15)\\ G_T &= k_T^T P^T + k_T^T R K_T + g_x^T + g_u^T K_T + k_T^T B^T V_{T+1}(A+BK_T) + G_{T+1}(A+BK_T) && (2.7.16)\\ W_T &= \frac{1}{2} k_T^T R k_T + g_u^T k_T + c + G_{T+1} B k_T + \frac{1}{2} k_T^T B^T V_{T-1} B k_T + W_{T-1} && (2.7.17) \end{aligned}$$

We find the control policy by minimizing $J_T$ with respect to $v_T$.

$$\begin{aligned} v_T &= \min_{v_T} c_T + J_{T+1} && (2.7.18)\\ &= z_T^T P v_T + \frac{1}{2} v_T^T R v_T + g_u^T v_T + \frac{1}{2}(A z_T + B v_T)^T V_{T+1}(A z_T + B v_T) + G_{T+1}(A z_T + B v_T) && (2.7.19)\\ &= \left(z_T^T P + z_T^T A^T V_{T+1} B\right) v_T + (G_{T+1} B + g_u^T) v_T + \frac{1}{2} v_T^T \left(R + B^T V_{T+1} B\right) v_T && (2.7.20)\\ & && (2.7.21) \end{aligned}$$

# Strategy: Build up on LQR




$$x_{t+1} = \frac{\partial f}{\partial x}\bigg|_{x_t} \delta x_t + \frac{\partial f}{\partial u}\bigg|_{u_t} \delta u_t + f(x_t^*, u_t^*)$$

$$x_{t+1} = A_t x_t + B_t u_t + x_t^{off}$$

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# Strategy: Build up on LQR

Iterative LQR

Affine LQR

Time-varying LQR

LQR

$$x_{t+1} = \frac{\partial f}{\partial x}\bigg|_{x_t} \delta x_t + \frac{\partial f}{\partial u}\bigg|_{u_t} \delta u_t + f(x_t^*, u_t^*)$$

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Affine LQR

Time-varying LQR

LQR

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# The iLQR Algorithm

1. Propose some initial (feasible) trajectory $\{x_t, u_t\}_{t=0}^{T-1}$
2. Linearize the dynamics, $f$ about trajectory:

$$\left.\frac{\partial f}{\partial x}\right|_{x_t} = A_t, \quad \left.\frac{\partial f}{\partial u}\right|_{u_t} = B_t$$

   Linearization can be obtained by three methods:

   (a) Analytical: either manually or via *auto-diff*, compute the correct derivatives.

   (b) Numerical: use finite differencing.

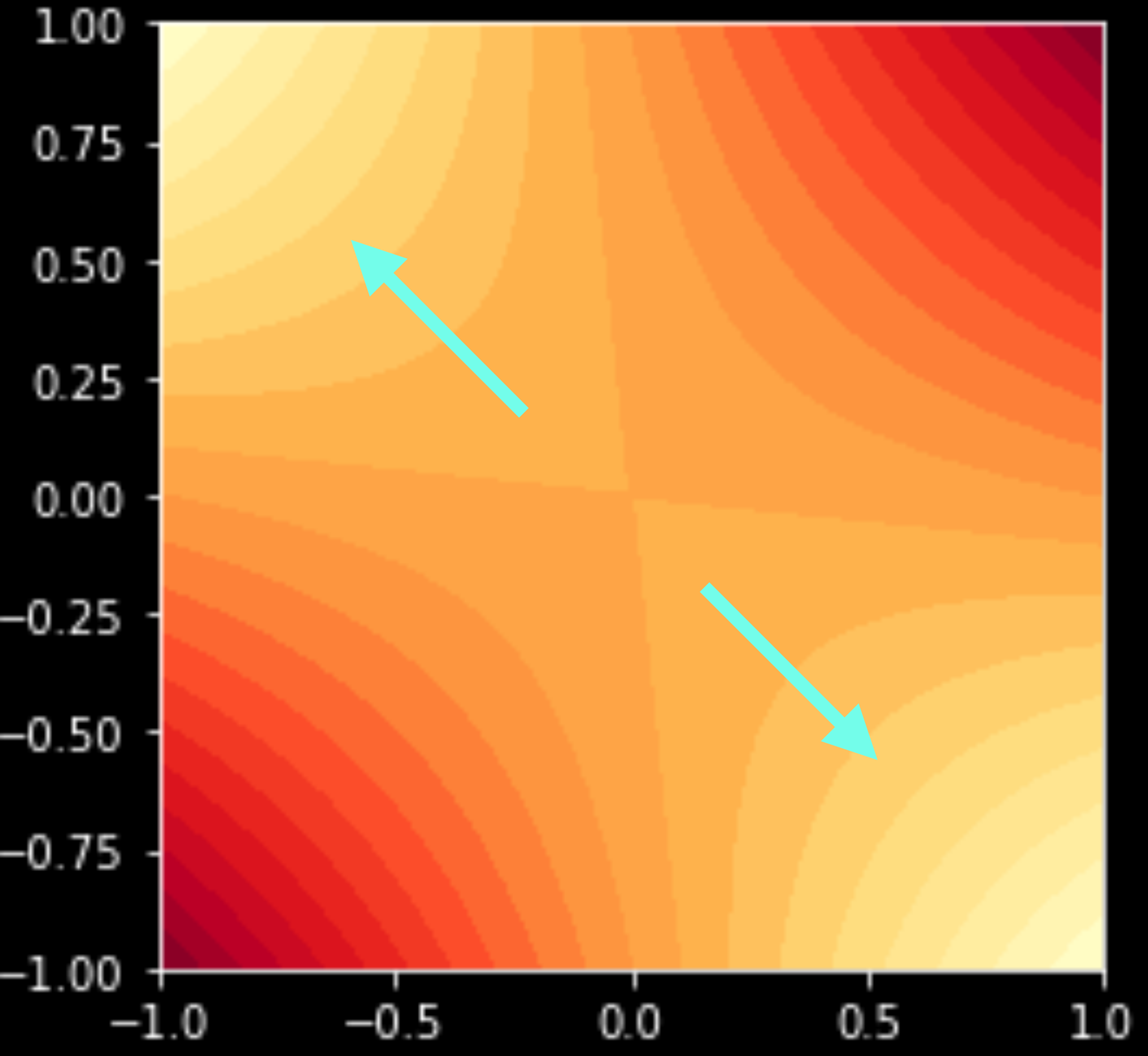
   (c) Statistical: Collect samples by deviations around the trajectory and fit linear model.

3. Compute second order Taylor series expansion the cost function $c(x, u)$ around $x_t$ and $u_t$ and get a quadratic approximation $c_t(\tilde{x}_t, \tilde{u}_t) = \tilde{x}_t^\top \tilde{Q}_t \tilde{x}_t + \tilde{u}_t^\top \tilde{R}_t \tilde{u}_t$ where the $\tilde{x}_t, \tilde{u}_t$ variables represent *changes* in the proposed trajectory in homogenous coordinates. [12]
4. Given $\{A_t, B_t, \tilde{Q}_t, \tilde{R}_t\}_{t=0}^{T-1}$, solve an affine quadratic control problem and obtain the proposed feedback matrices (on the homogeneous represenation of $x$).
5. Forward simulate the full nonlinear model $f(x, u)$ using the computed controls $\{u_t\}_{t=0}^{T-1}$ that arise from feedback matrices applied to the sequence of states $\{x_t\}_{t=0}^{T-1}$ that arise from that forward simulation.
6. Using the newly obtained $\{x_t, u_t\}_{t=0}^{T-1}$ repeat steps from 2.

# Approximations always hurt

# #1: Q and R not PSD / PD

Quadracizing non-convex cost function



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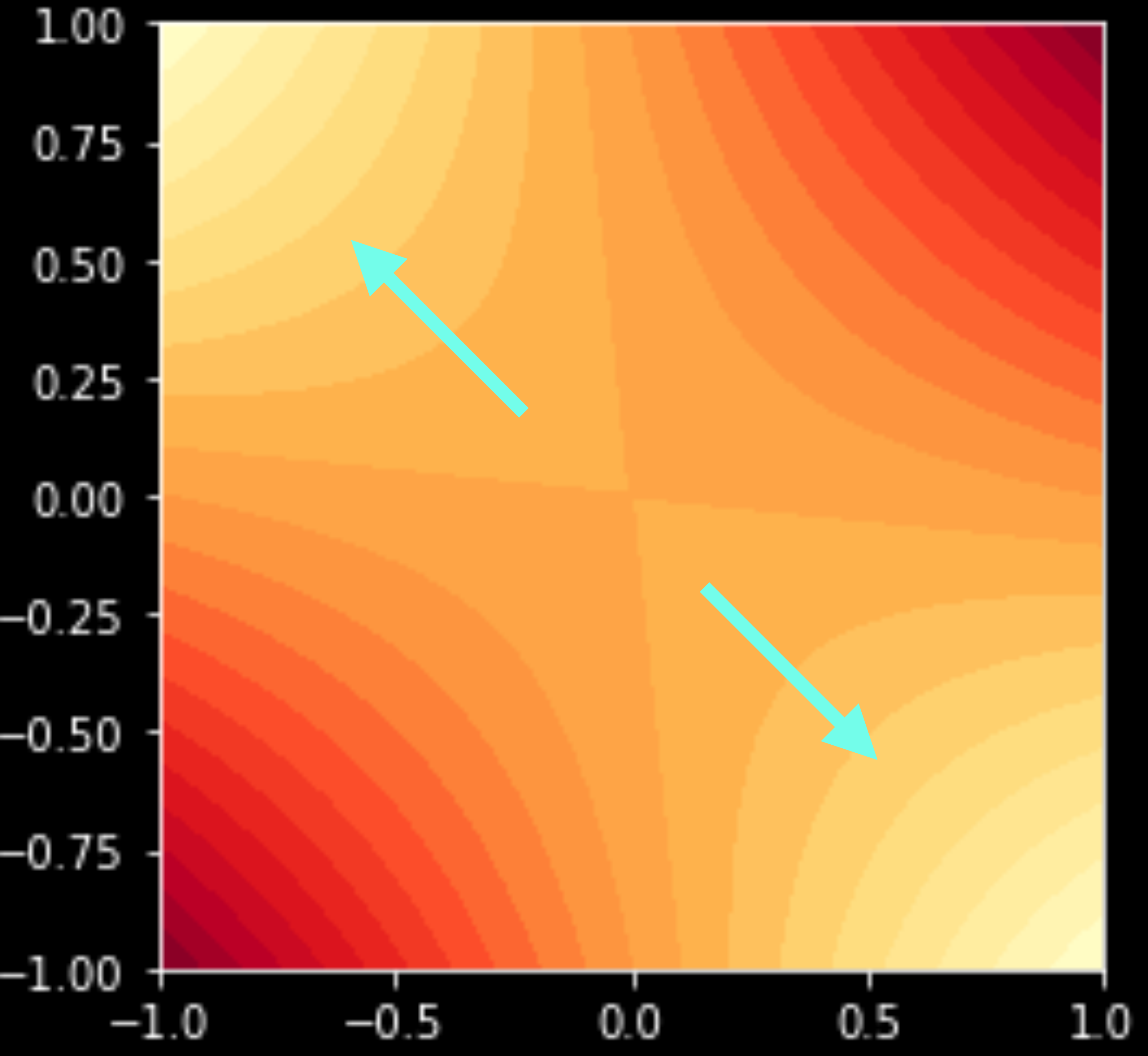
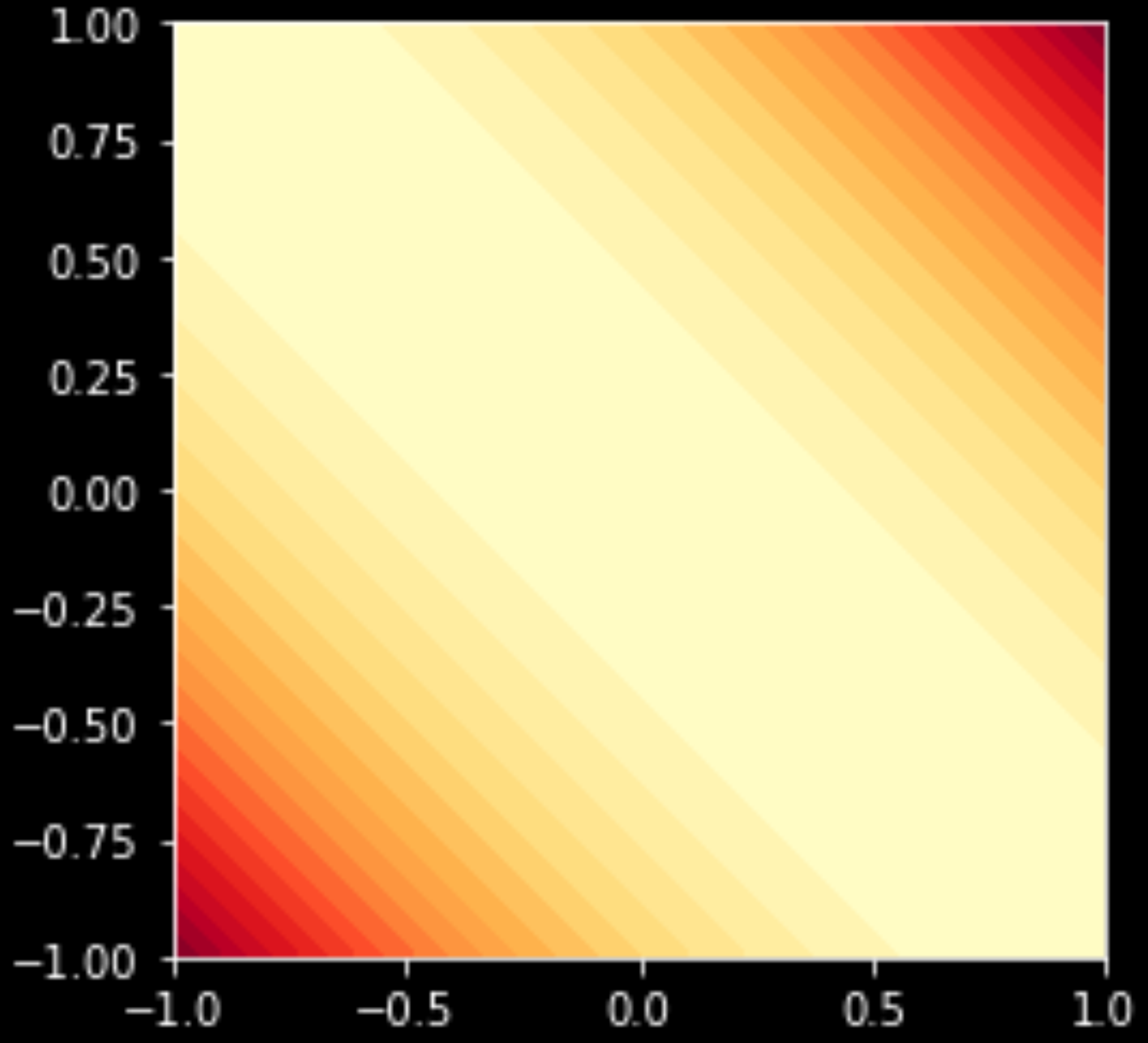



Eigen-value decomposition

$$Q = U\Sigma U^T$$

Set negative eigen values to 0

$$\Sigma = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \longrightarrow \Sigma = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$$



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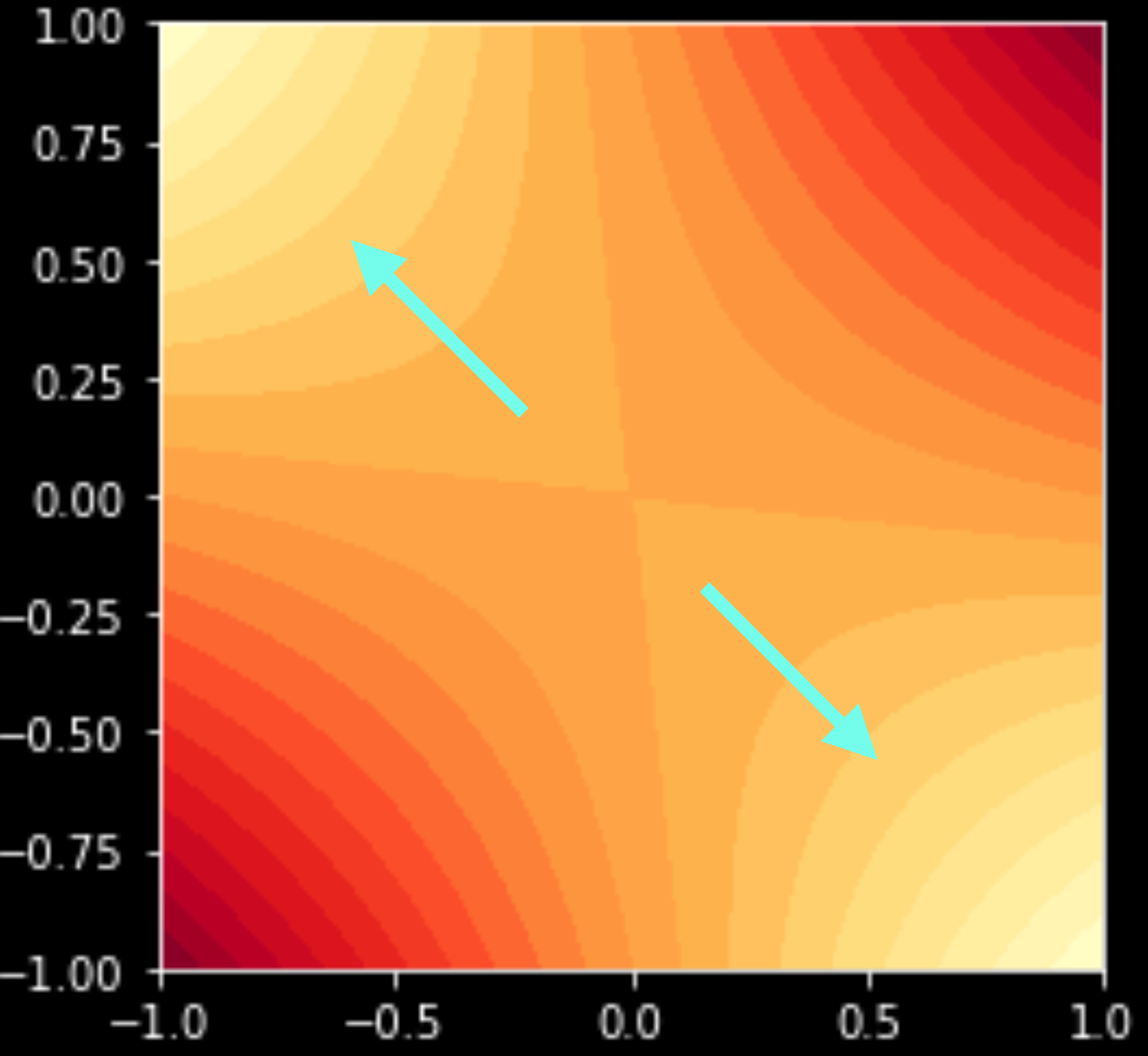
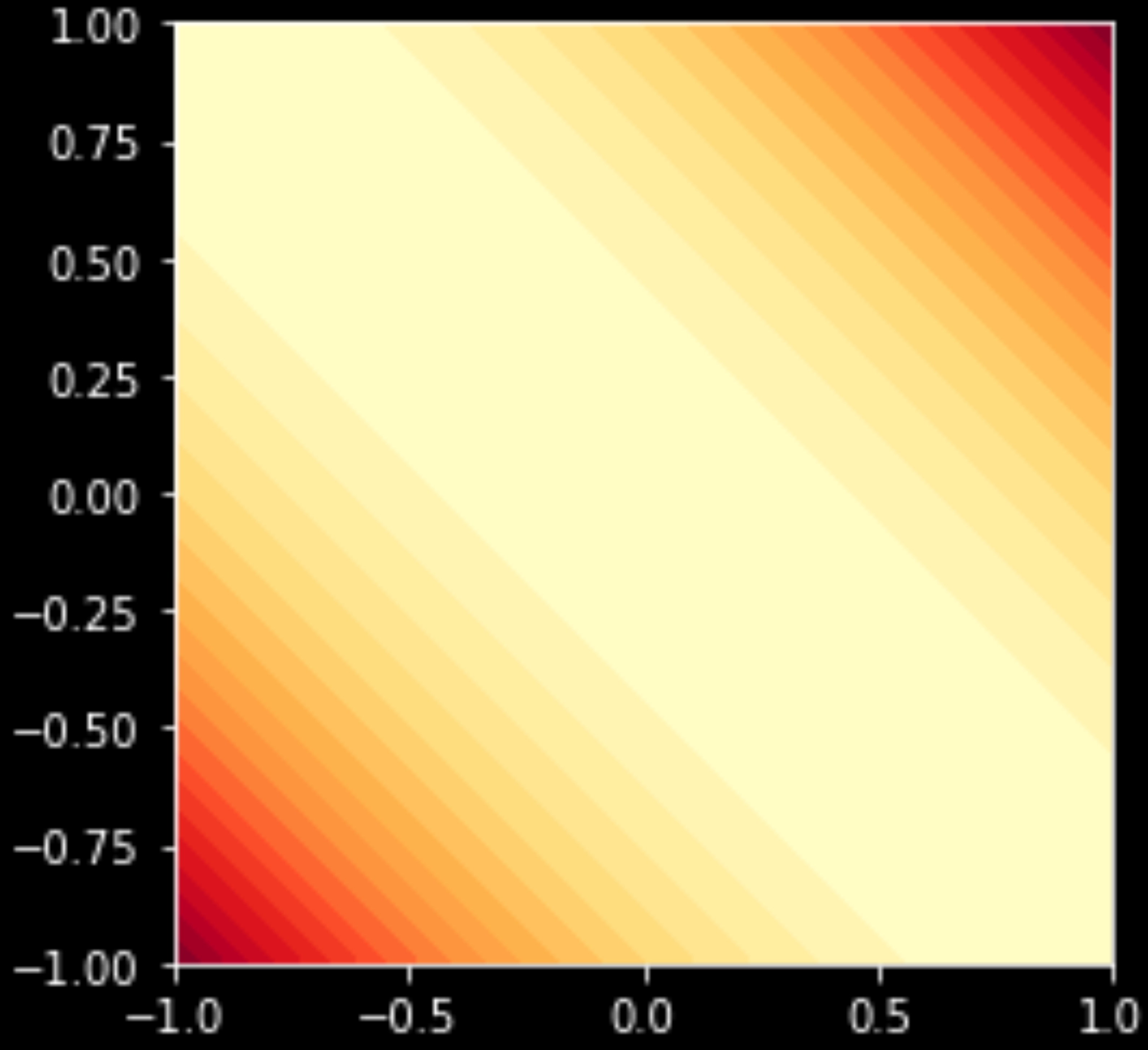


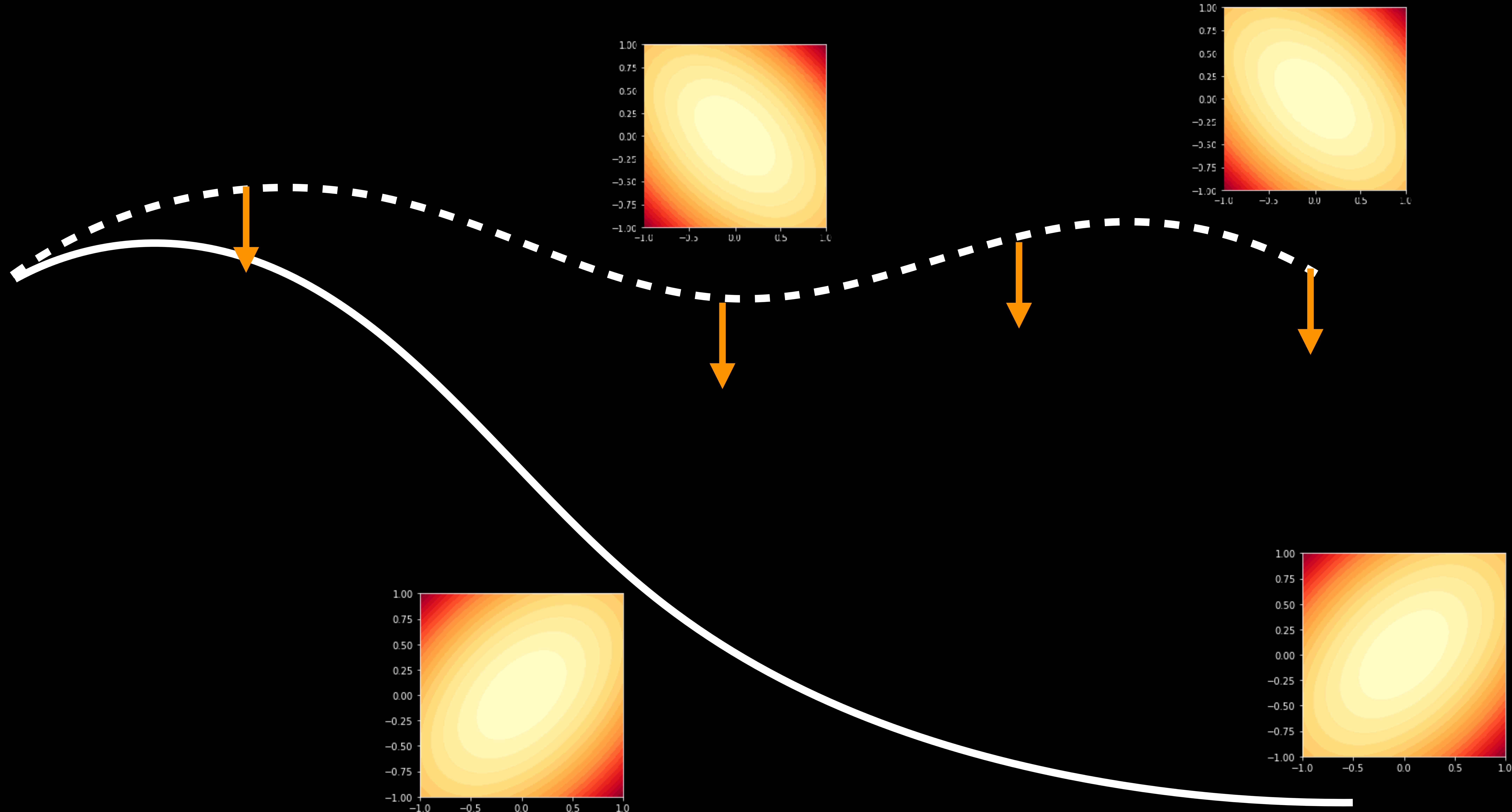

Increase diagonal values

$$Q = Q + \lambda I$$

$\lambda = 4$



# #2: Approximation Errors Compound



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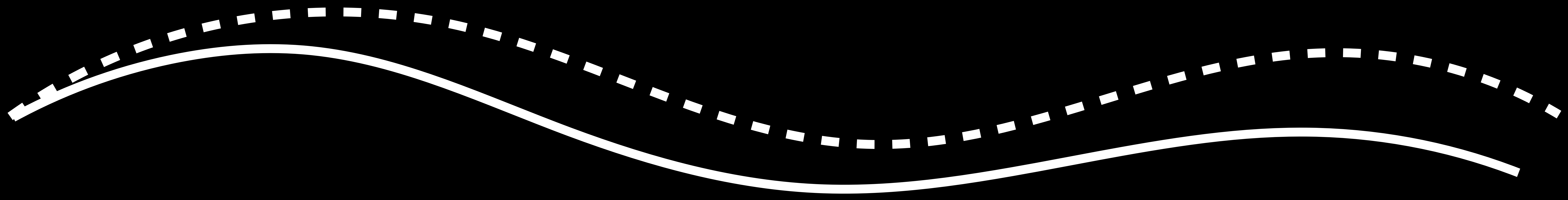

Slowly change controls

$$u = (1 - \alpha)u_{old} + \alpha u_{new}$$

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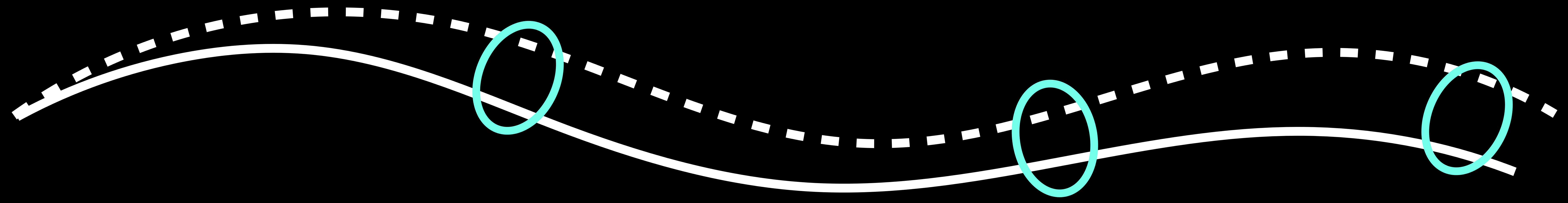


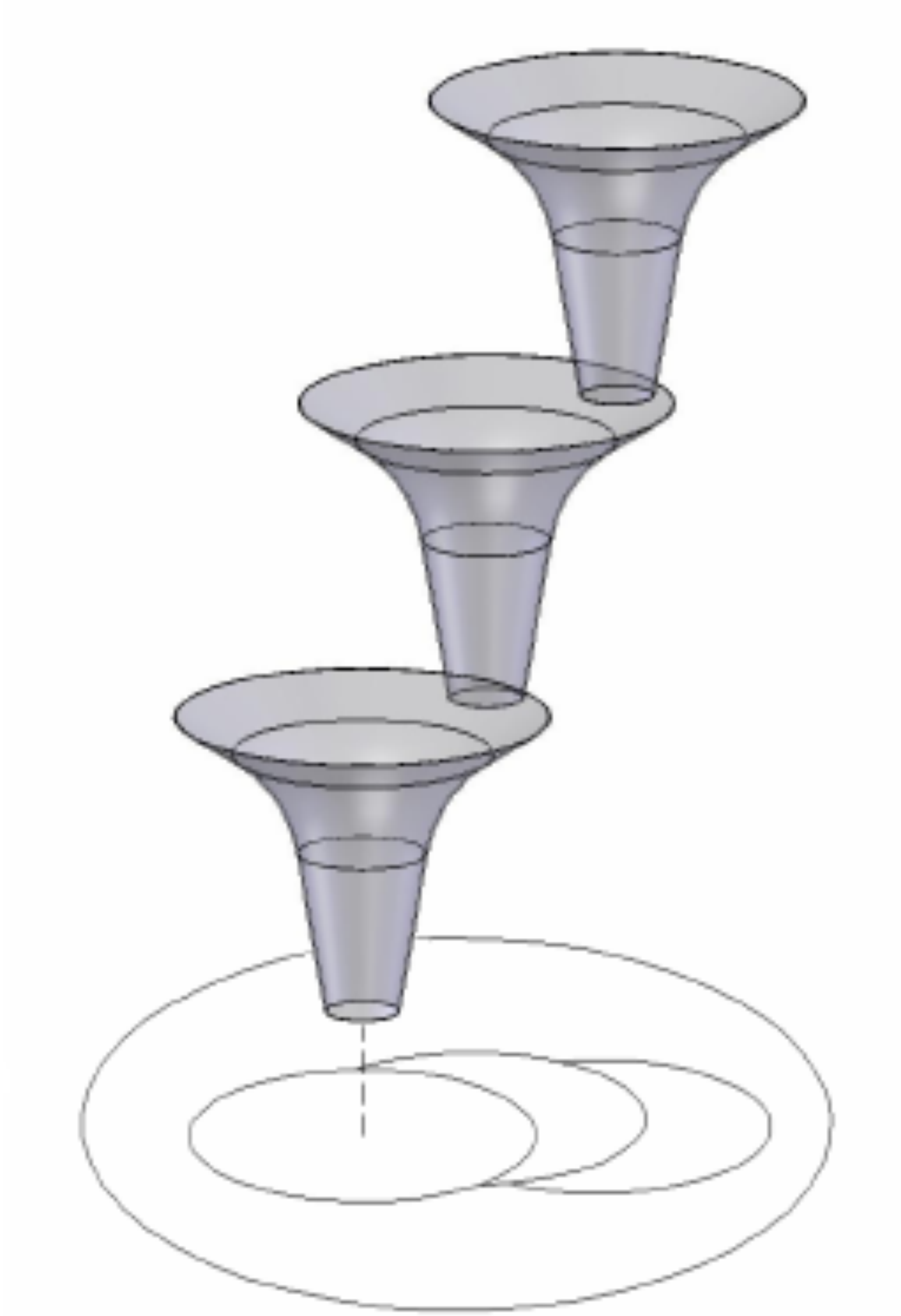
Trust region: Control and state sampling

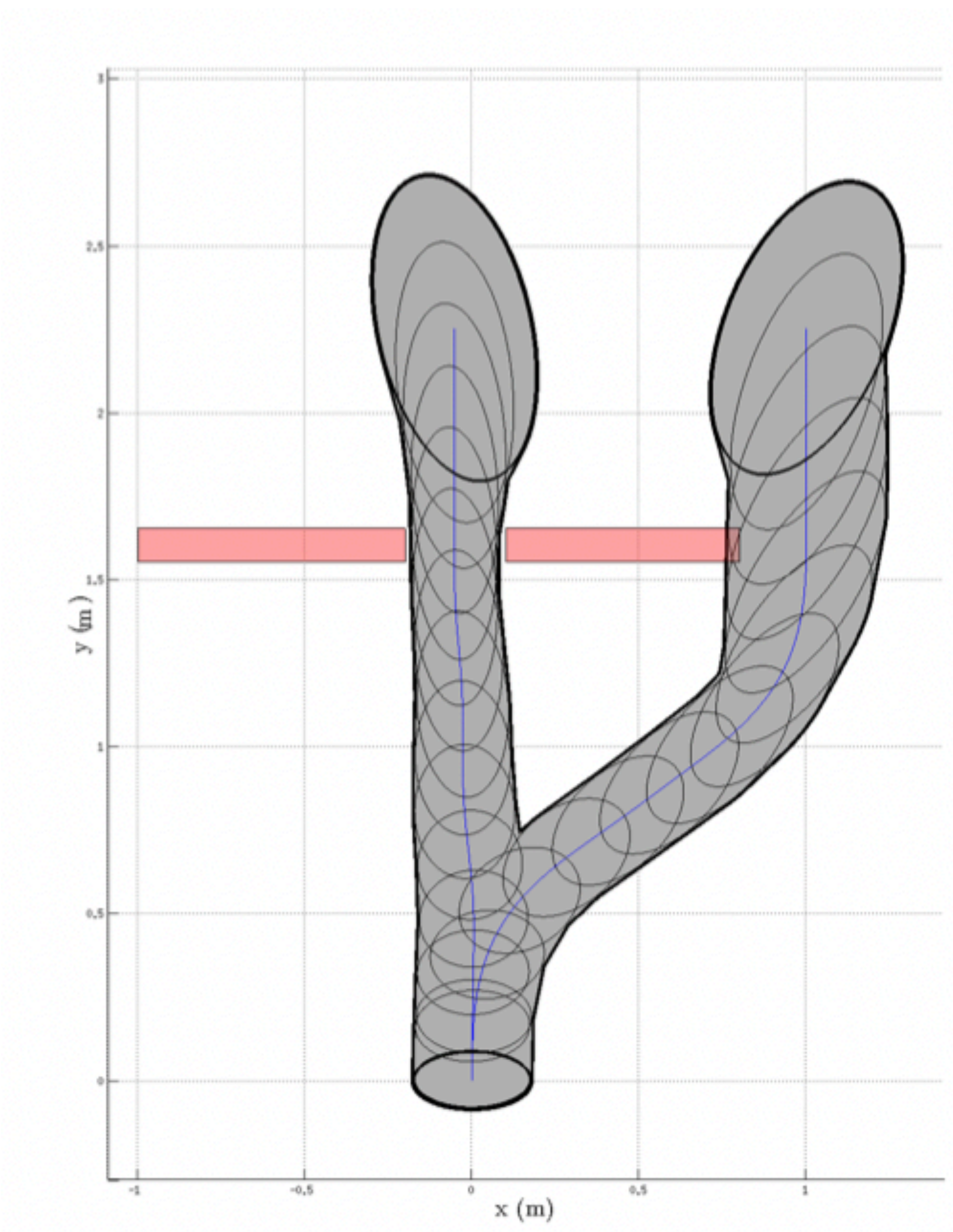
$$c_{new}(x,u) = c(x,u) + \lambda_x ||x - x_{old}|| + \lambda_u ||u - u_{old}||$$

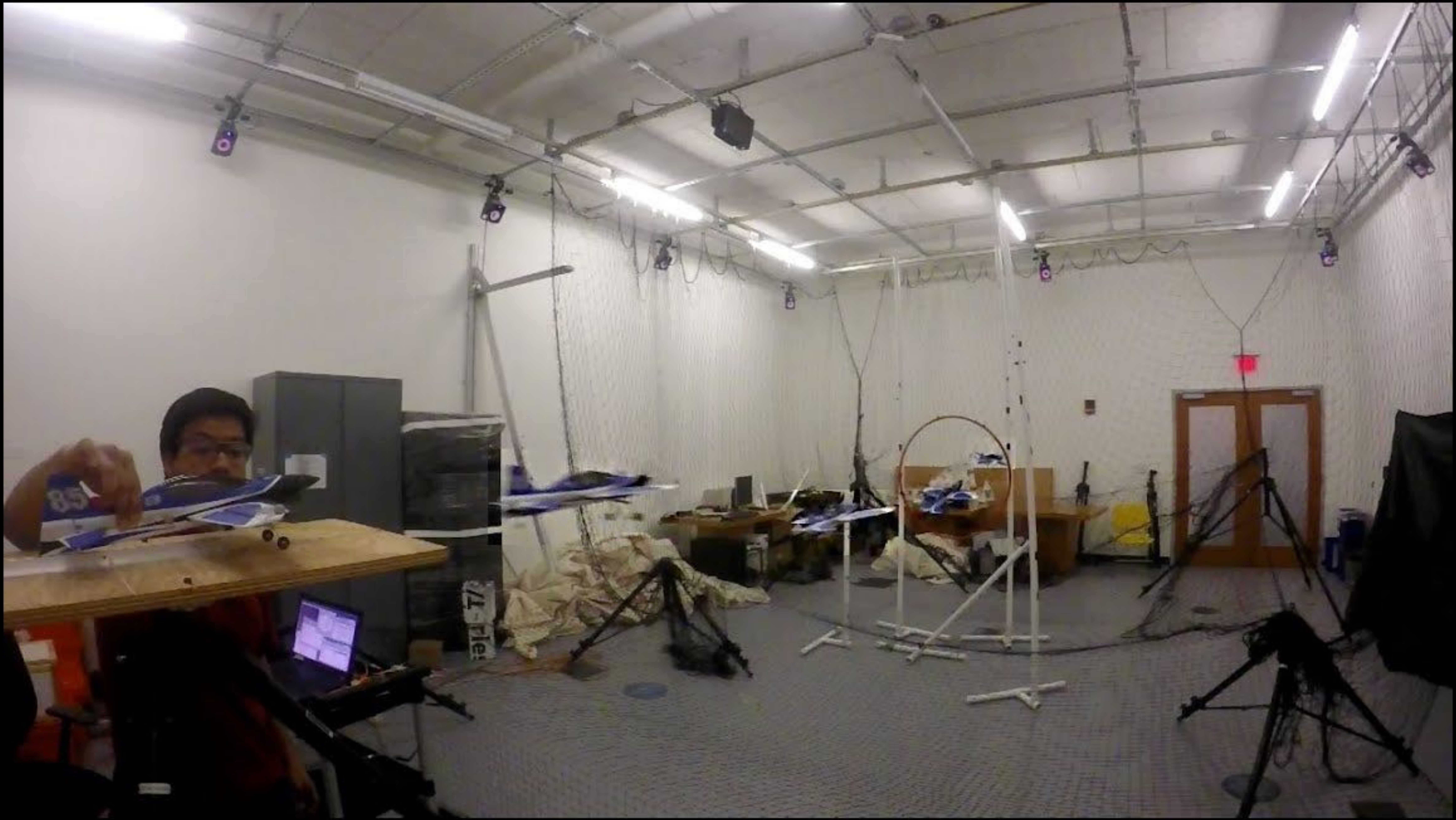
(Penalize deviations from old state / control)

# How general is this idea?

# #1: Cover the world with funnels



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# #2: Replace linear/quadratic with a LEARNER

for i = 1 ..... N

Roll-out current policy

~~Linearize dynamics,~~
~~Quadricize costs about traj~~

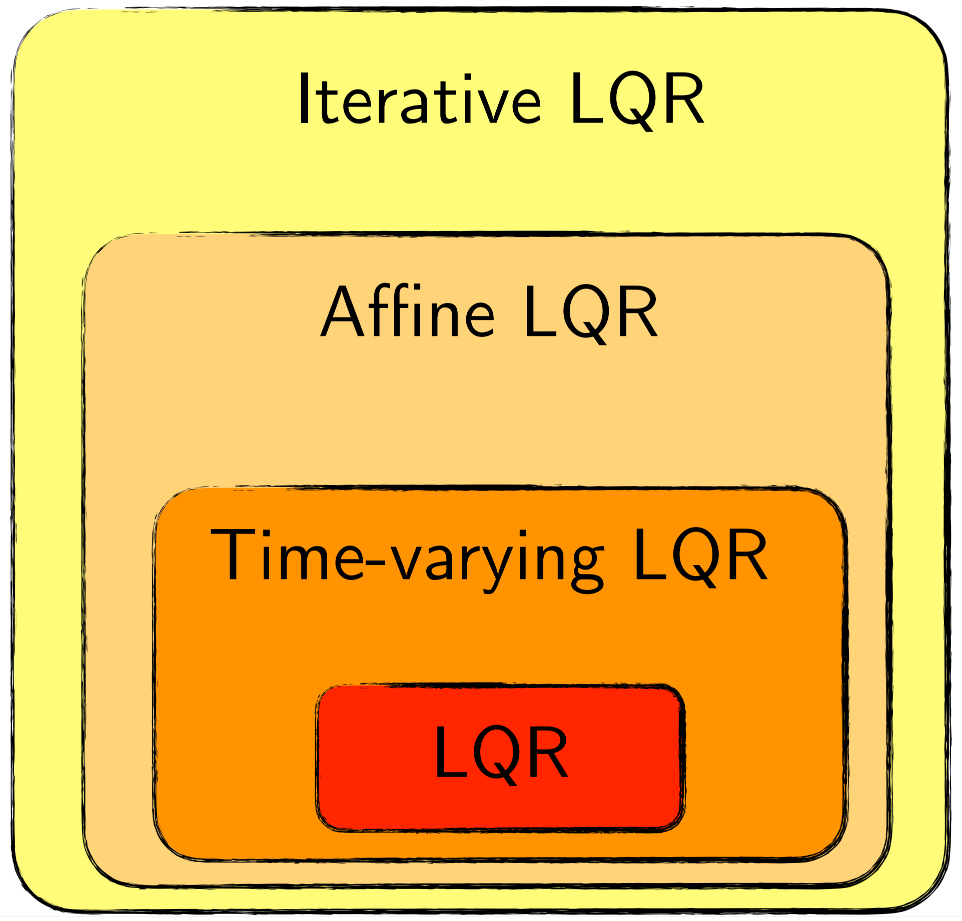
→ Train model from collected data!

Update policy

# tl;dr

LQR is
fundamentally a way
to
*locally approximate*
and
update value functions

## Strategy: Build up on LQR

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