Linear Quadratic Regulator:

The Analytic MDP

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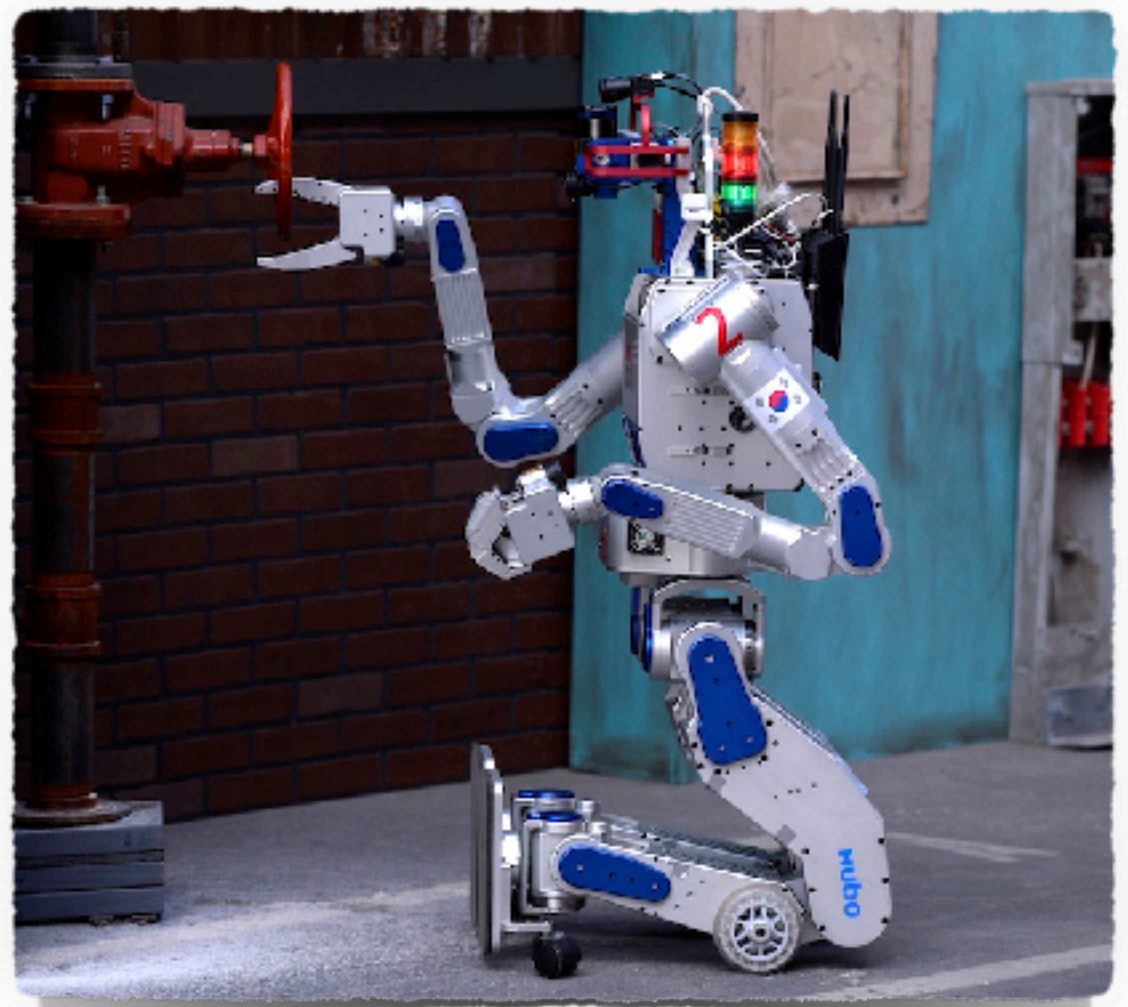
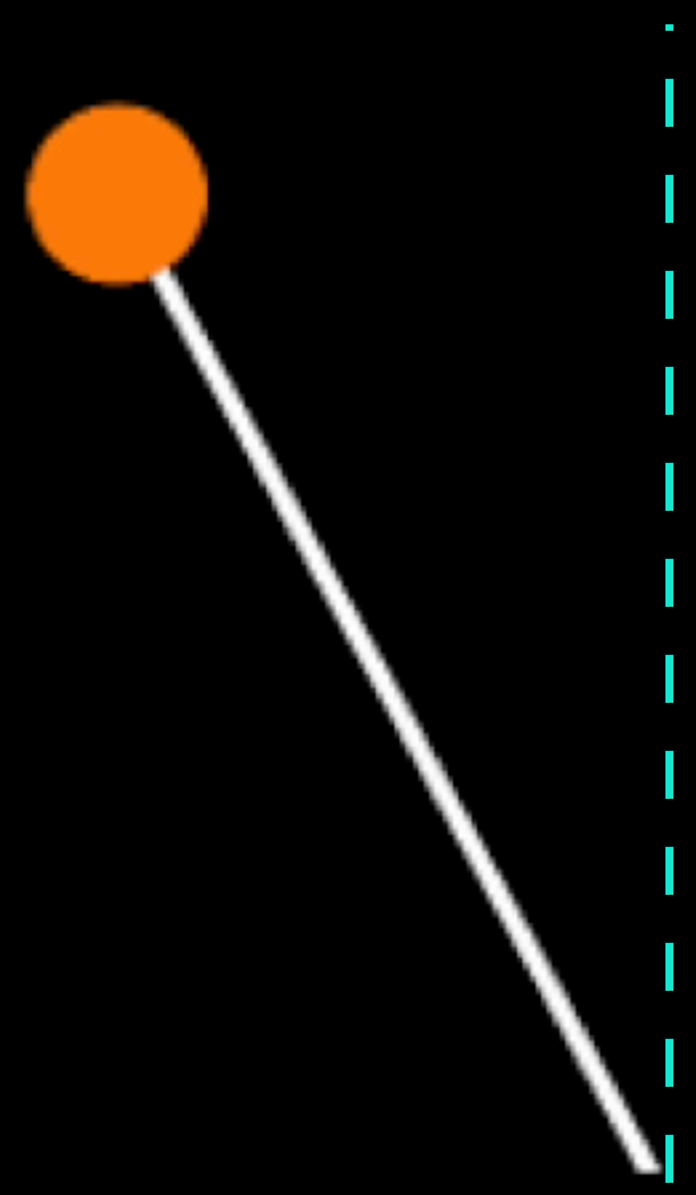
It's time to bring in the robots!

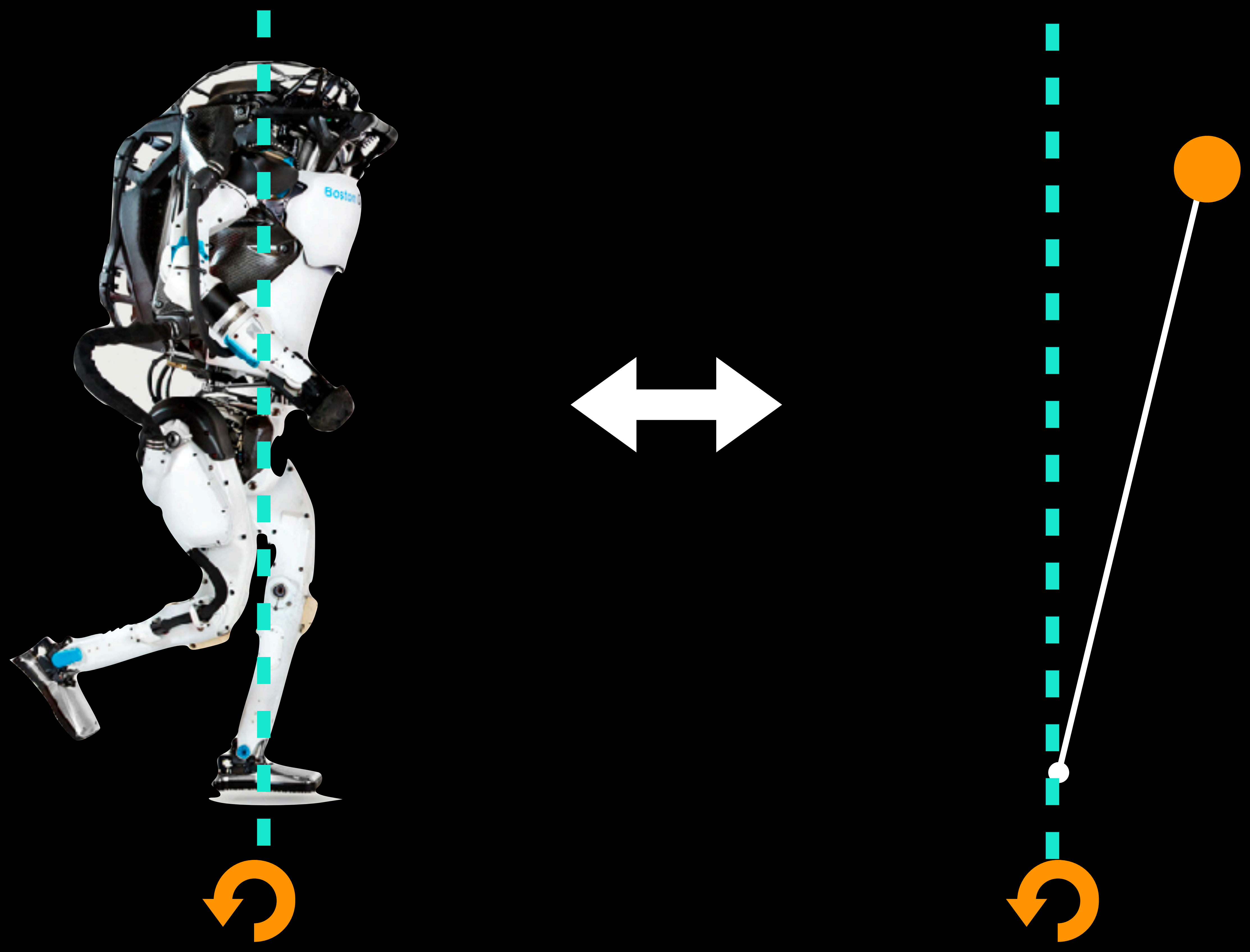



HD-02
BostonDynamics

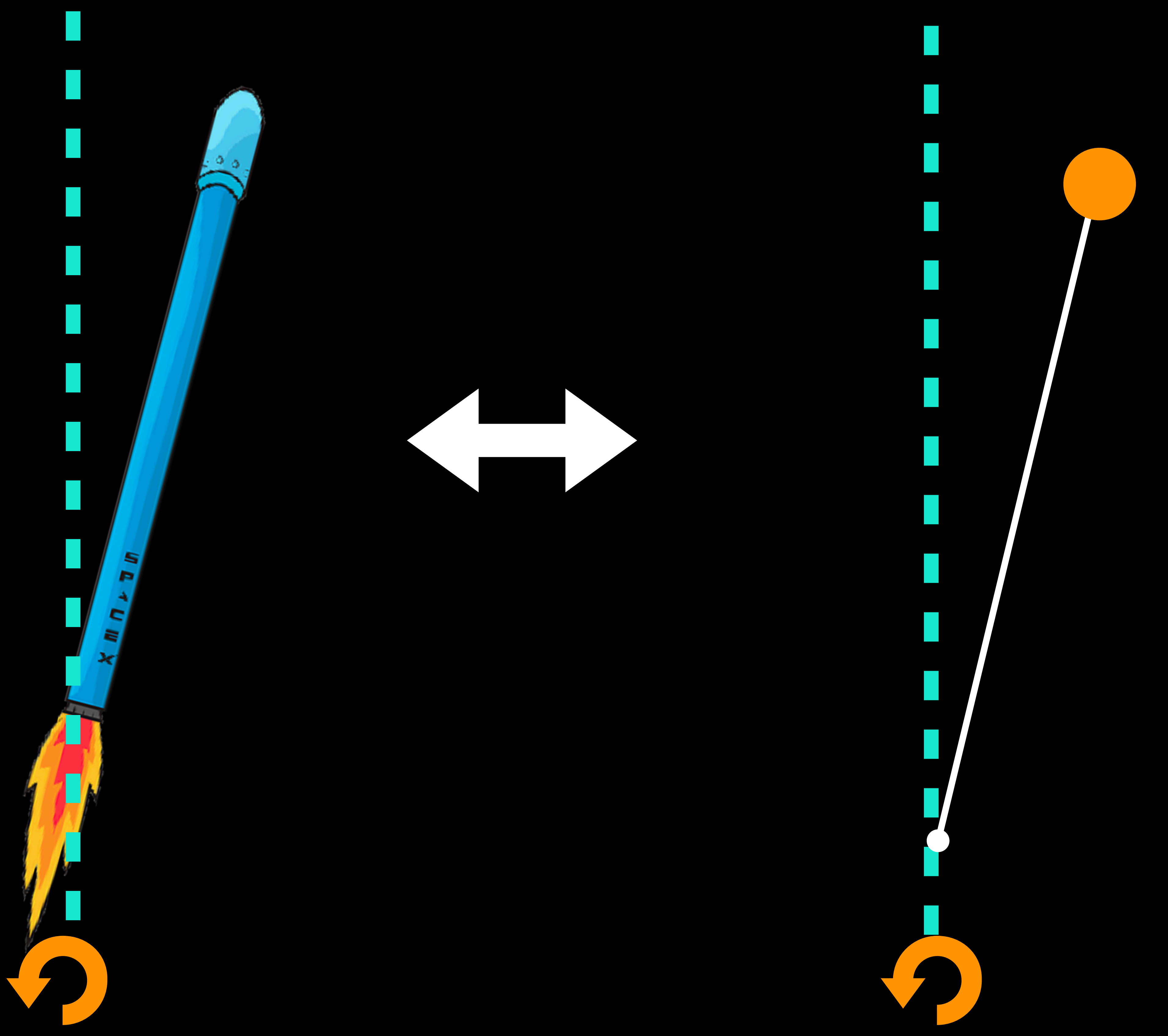
The Inverted Pendulum: A fundamental dynamics model



Humanoid balancing

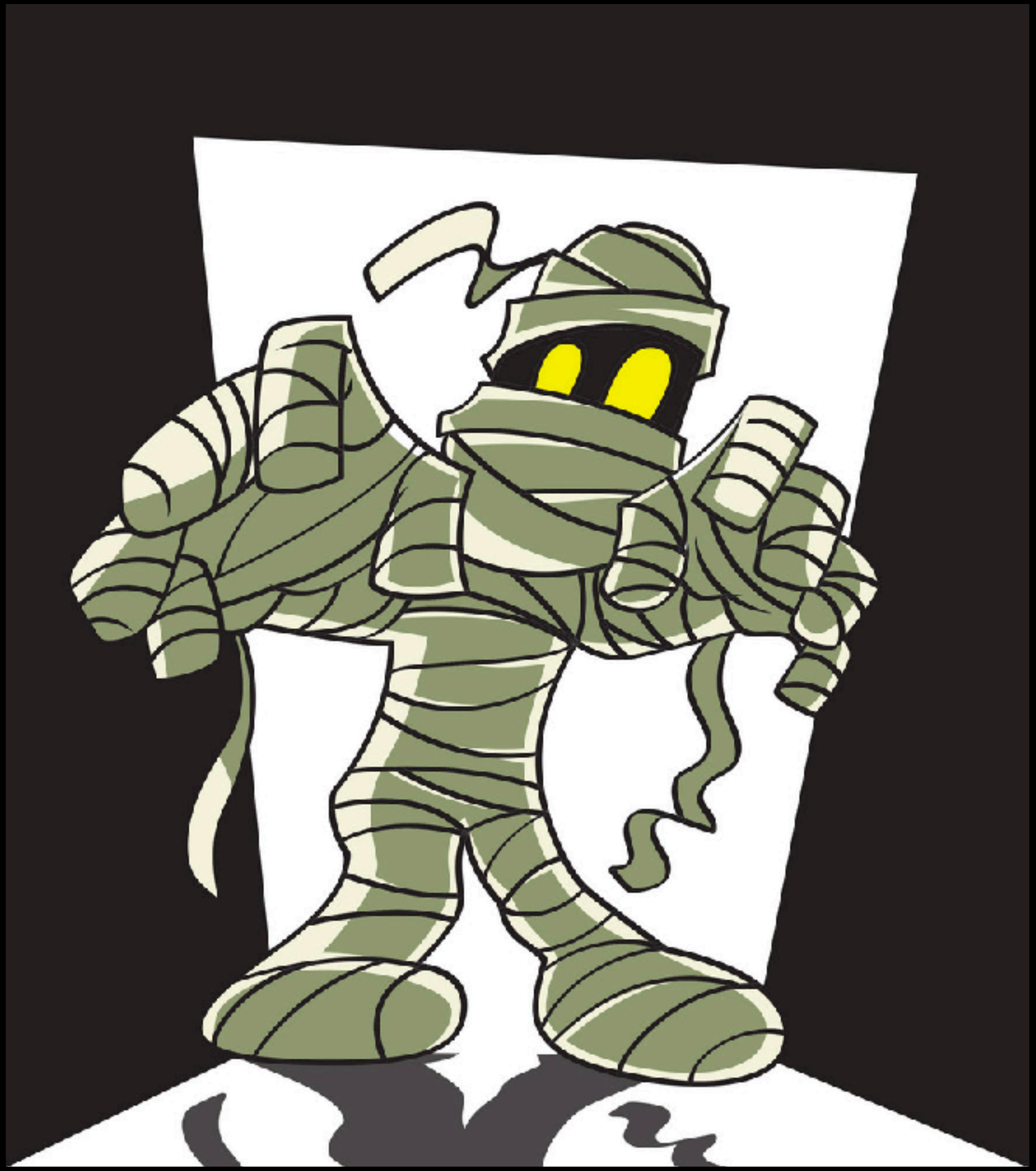


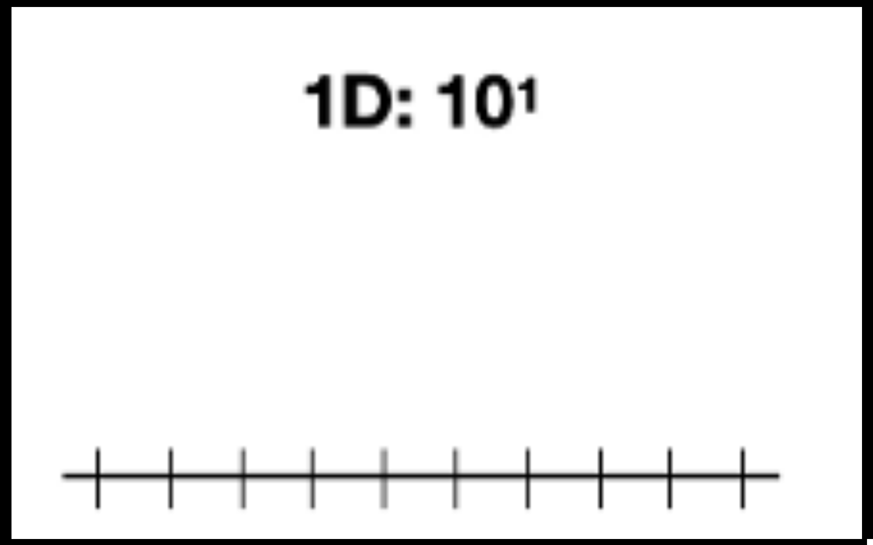
Rocket landing

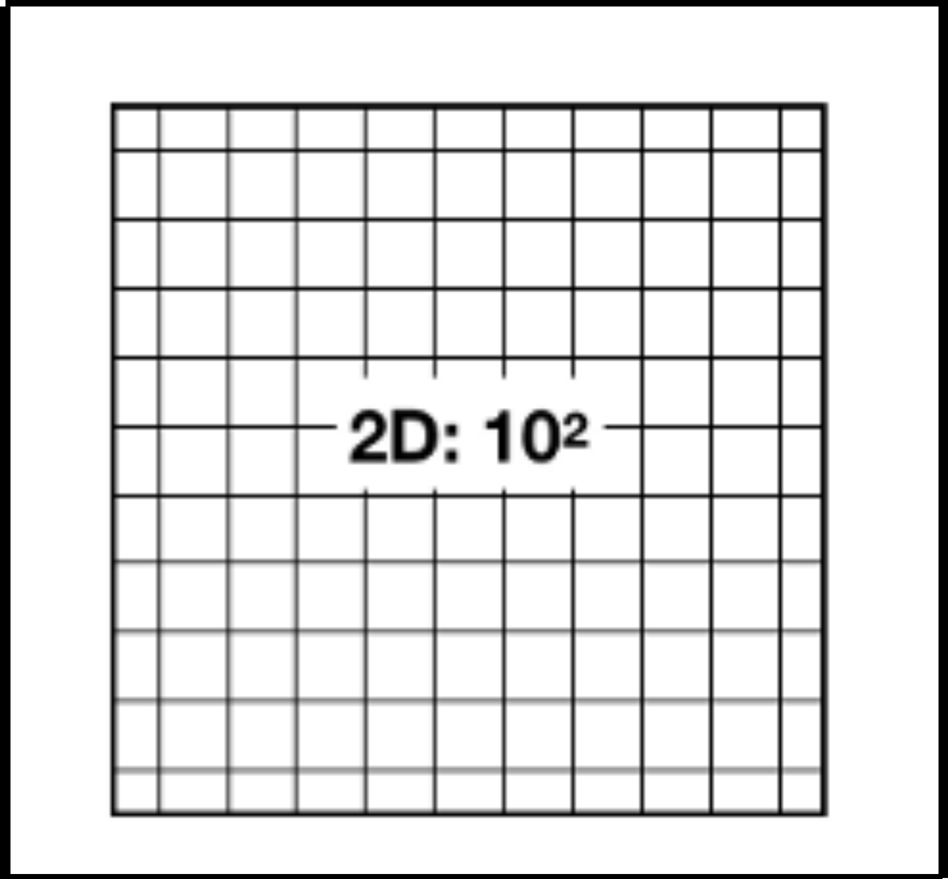


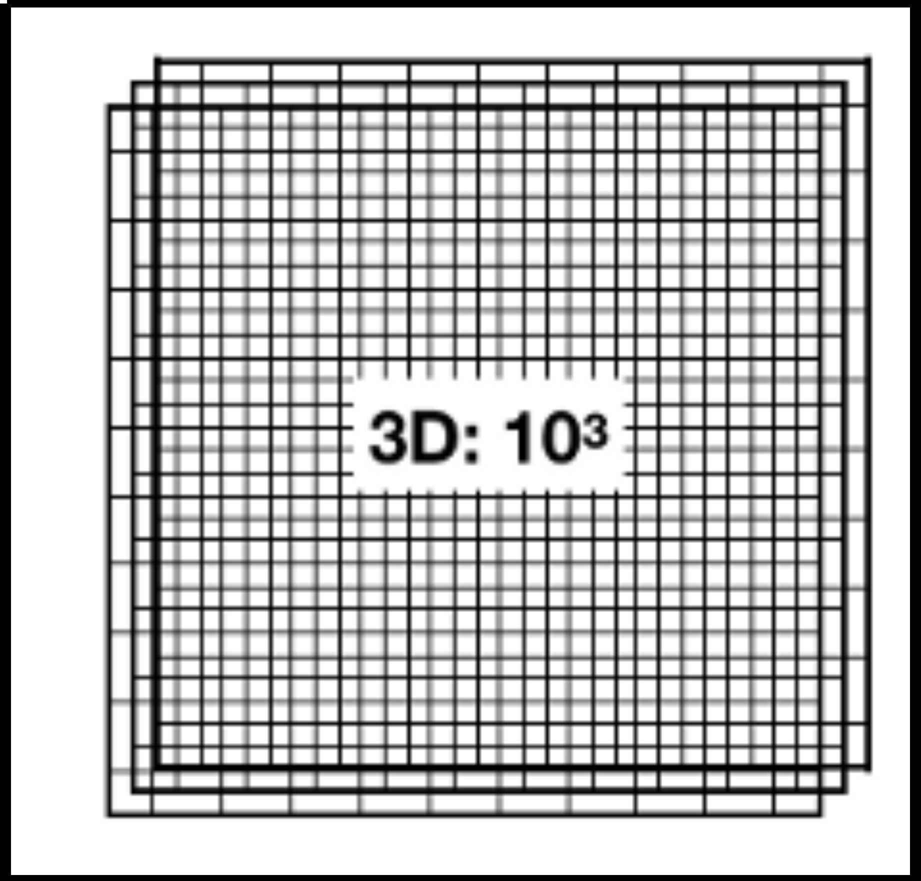
Why not discretize the dynamics and apply value / policy iteration?

THE CURSE OF DIMENSIONALITY





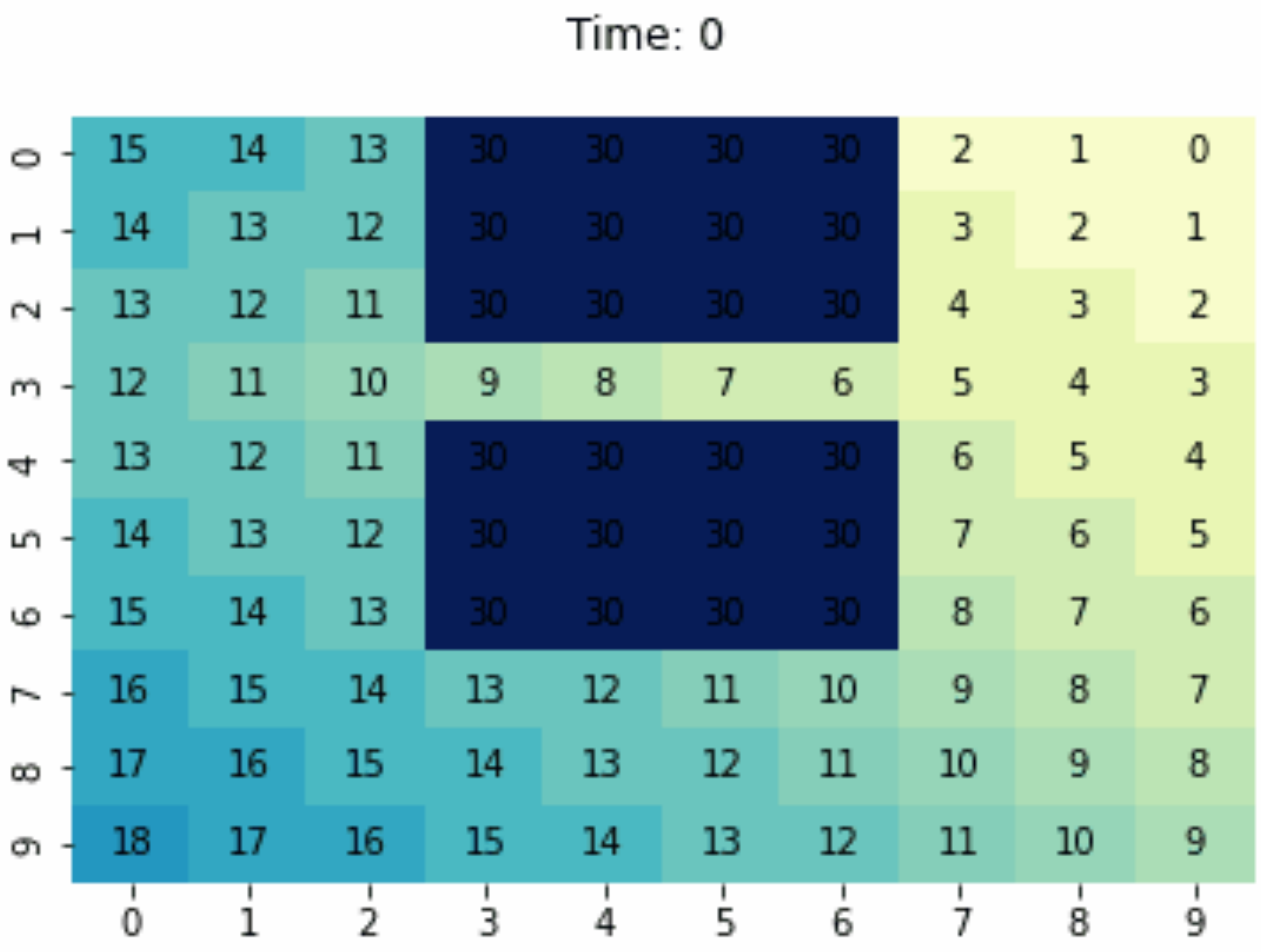




No Discretization!

Can we analytically *represent* and *update* the value function?

$$V^*(s) = \min_a \left[c(s,a) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^*(s')\right]$$



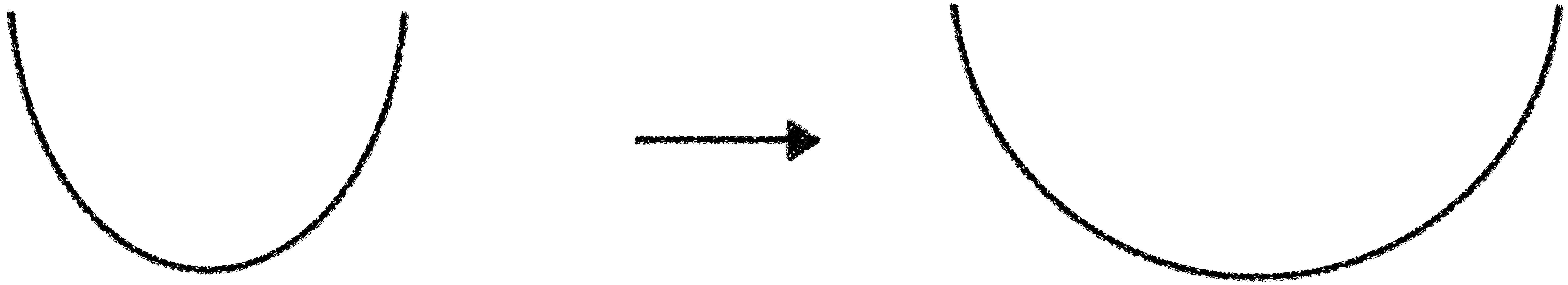
$$V^*(s) = \min_a \left[c(s,a) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^*(s)\right]$$

Can represent analytically ...
(piecewise linear?)

But updating seems hard!

Can we analytically represent and *update* the value function?

Yes*



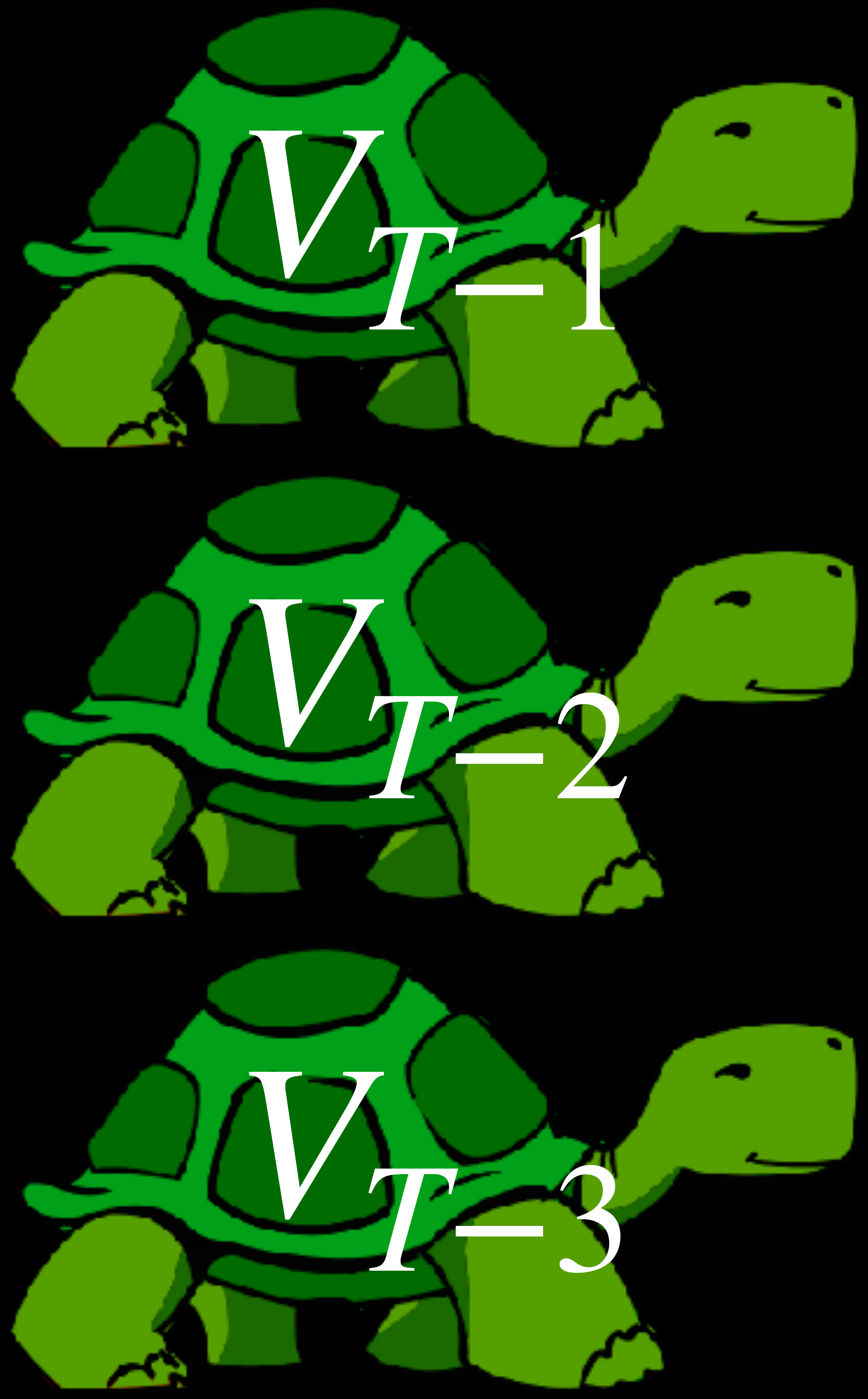
*linear dynamics, quadratic costs

Let’s formalize!

It’s quadratics all the way down!

$$K_t = (R + B^T V_{t+1} B)^{-1} B^T V_{t+1} A$$

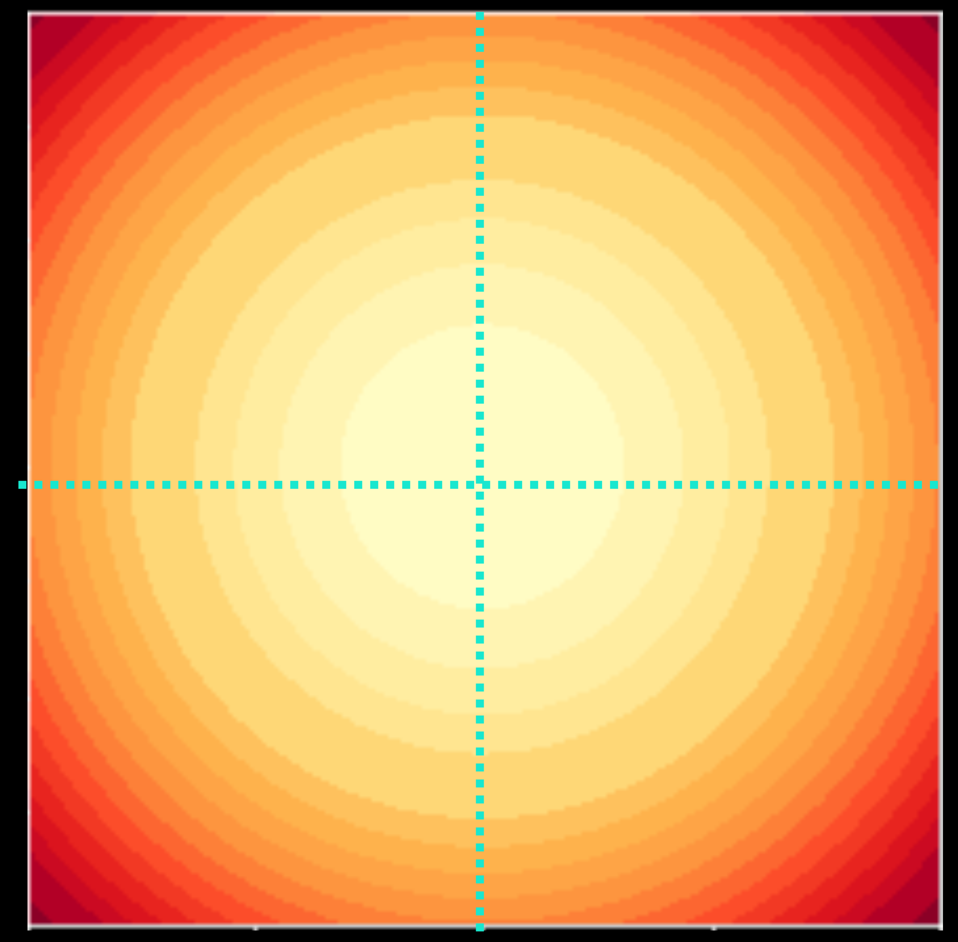
$$V_t = Q + K_t^T R K_t + (A + BK_t)^T V_{t+1} (A + BK_t)$$



The LQR Algorithm

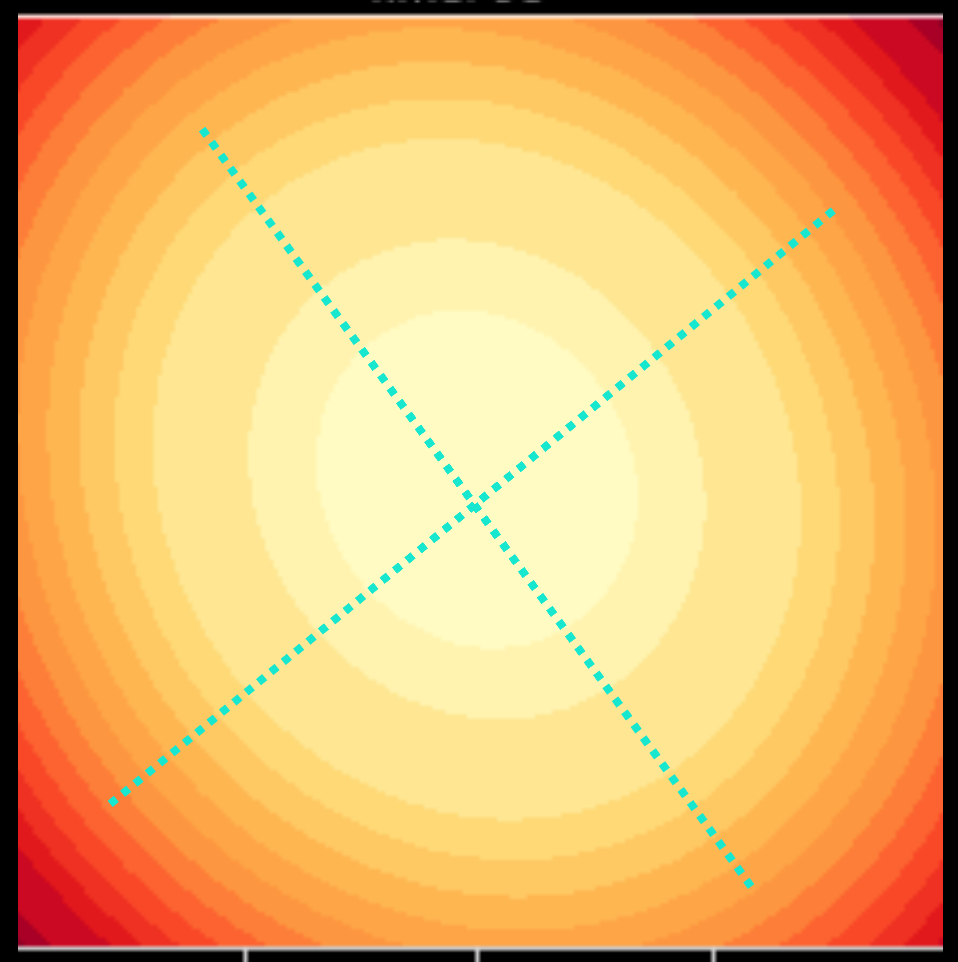
Initialize $V_T = Q$

For t = T … 1



Compute gain matrix

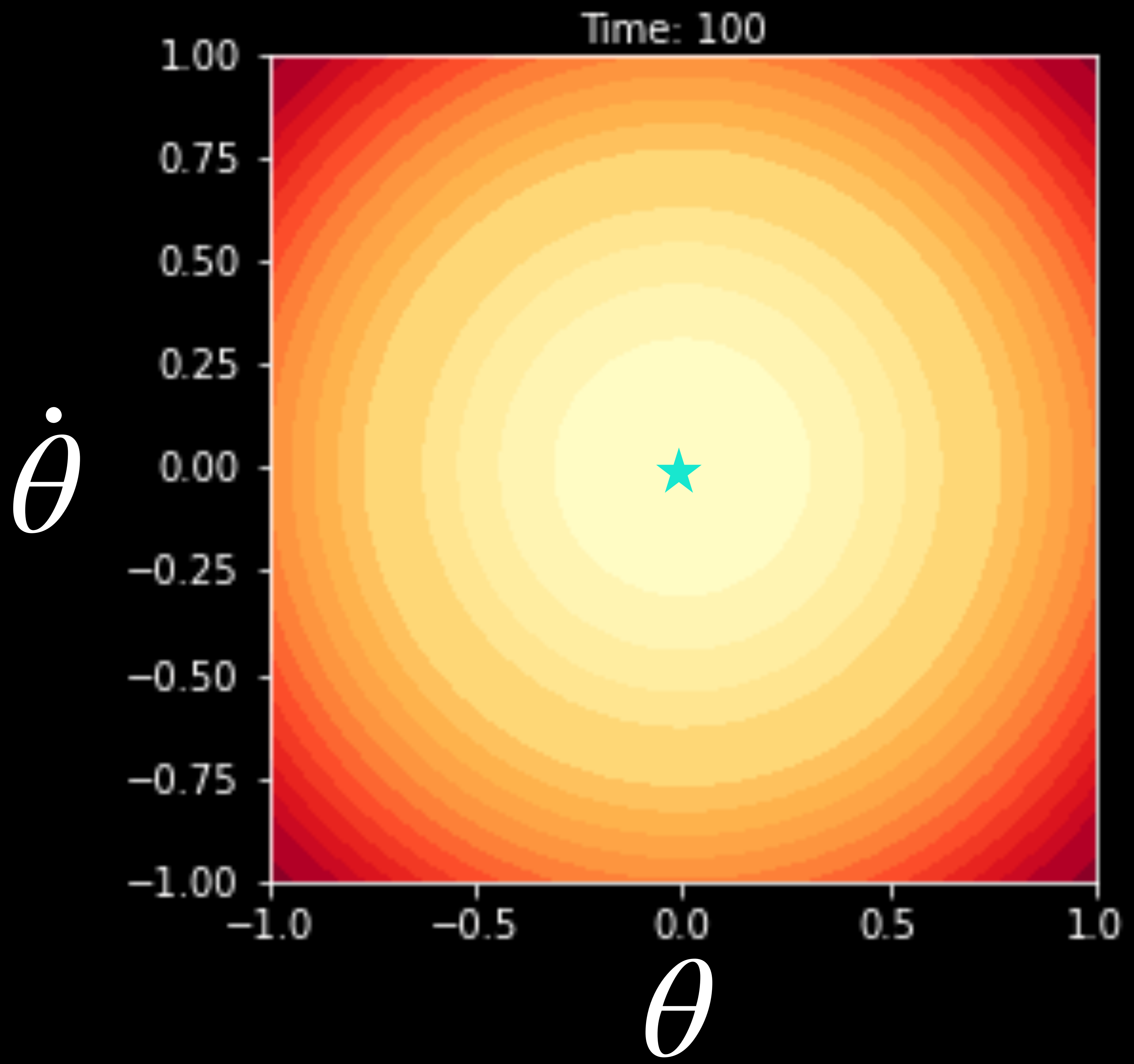
$$K_t = (R + B^T V_{t+1} B)^{-1} B^T V_{t+1} A$$



Update value

$$V_t = Q + K_t^T R K_t + (A + BK_t)^T V_{t+1} (A + BK_t)$$

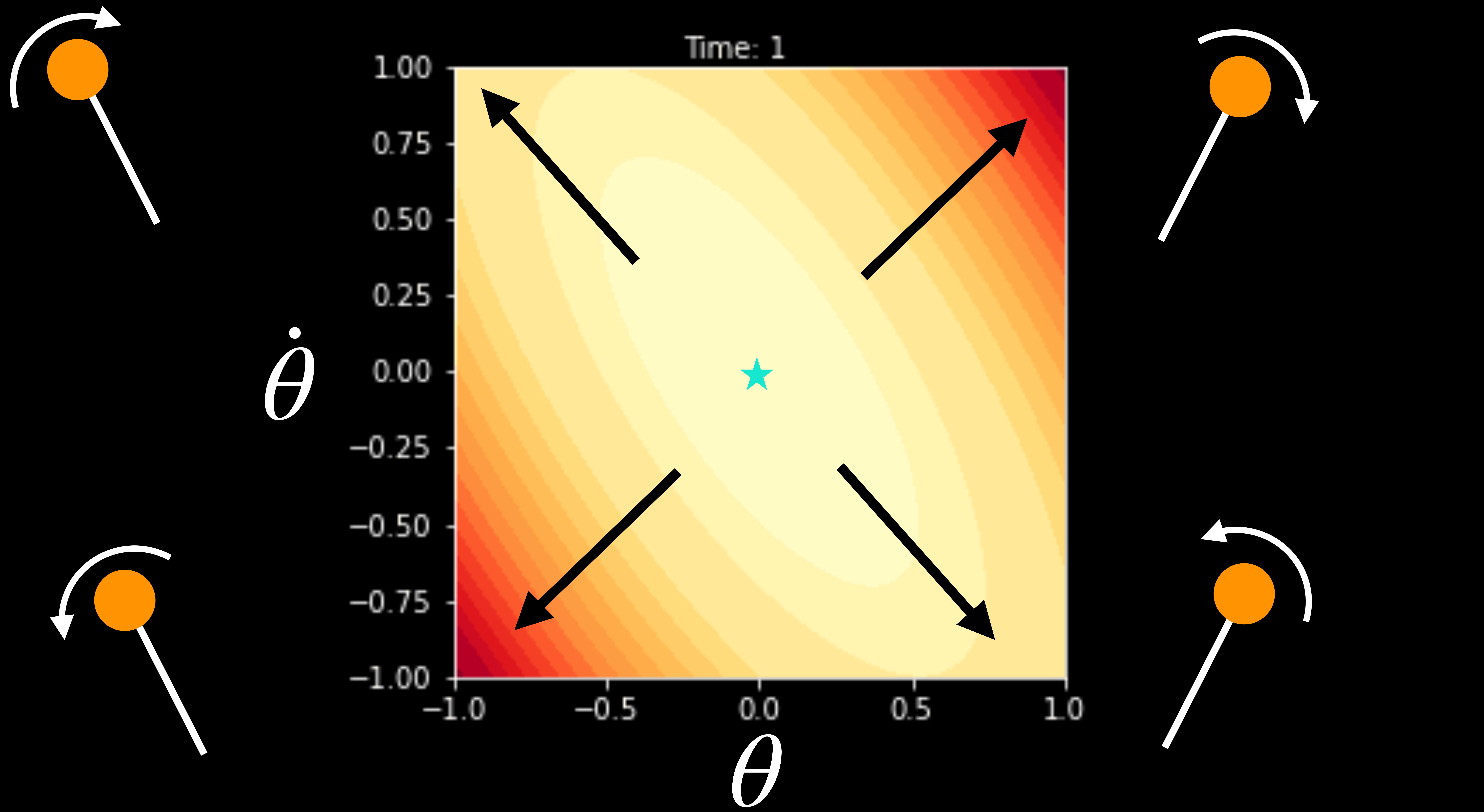
Value Iteration for Inverted Pendulum



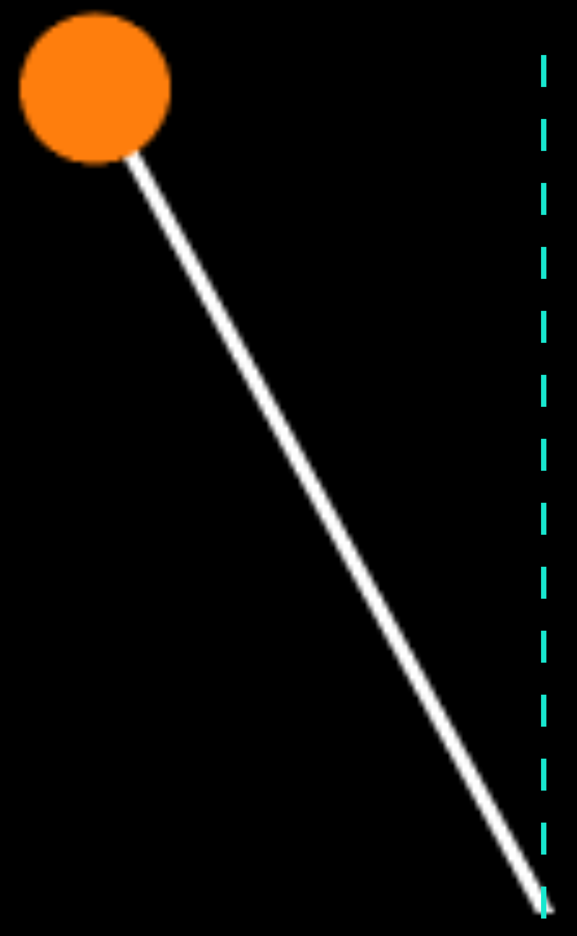
Value converges when system is stabilizable

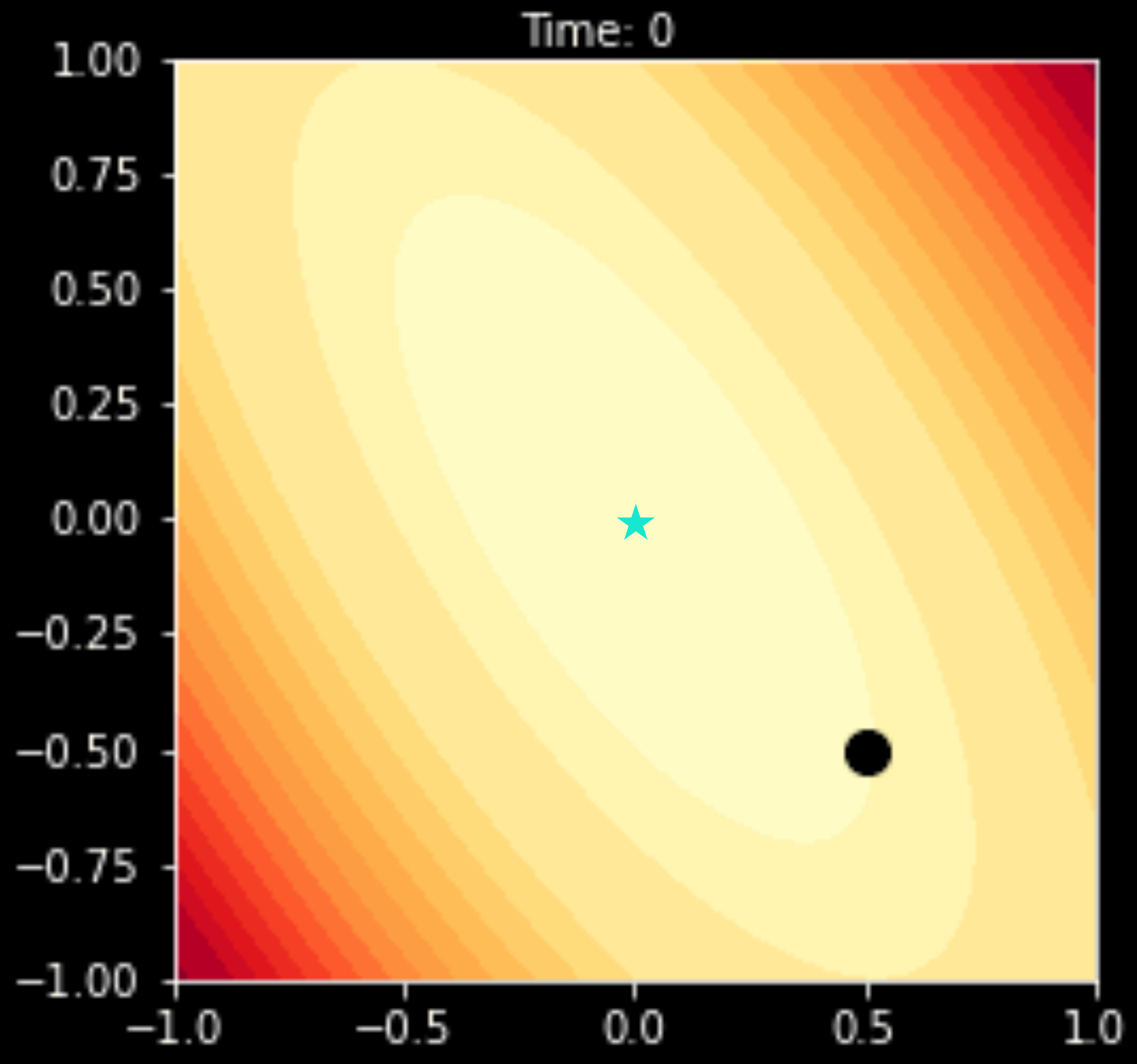
Can solve Ricatti equations for fixed point

Value Iteration for Inverted Pendulum



An Easy Starting Point





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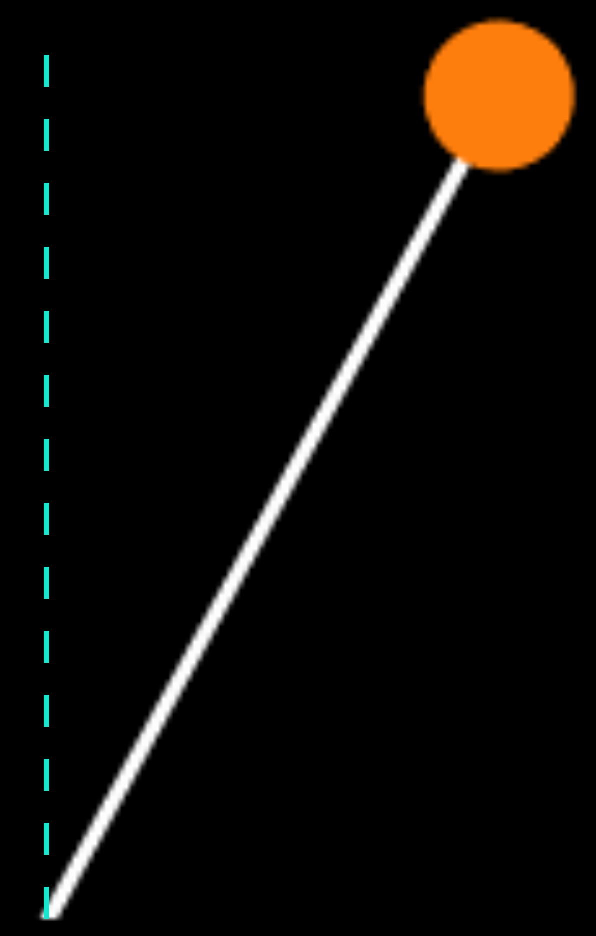
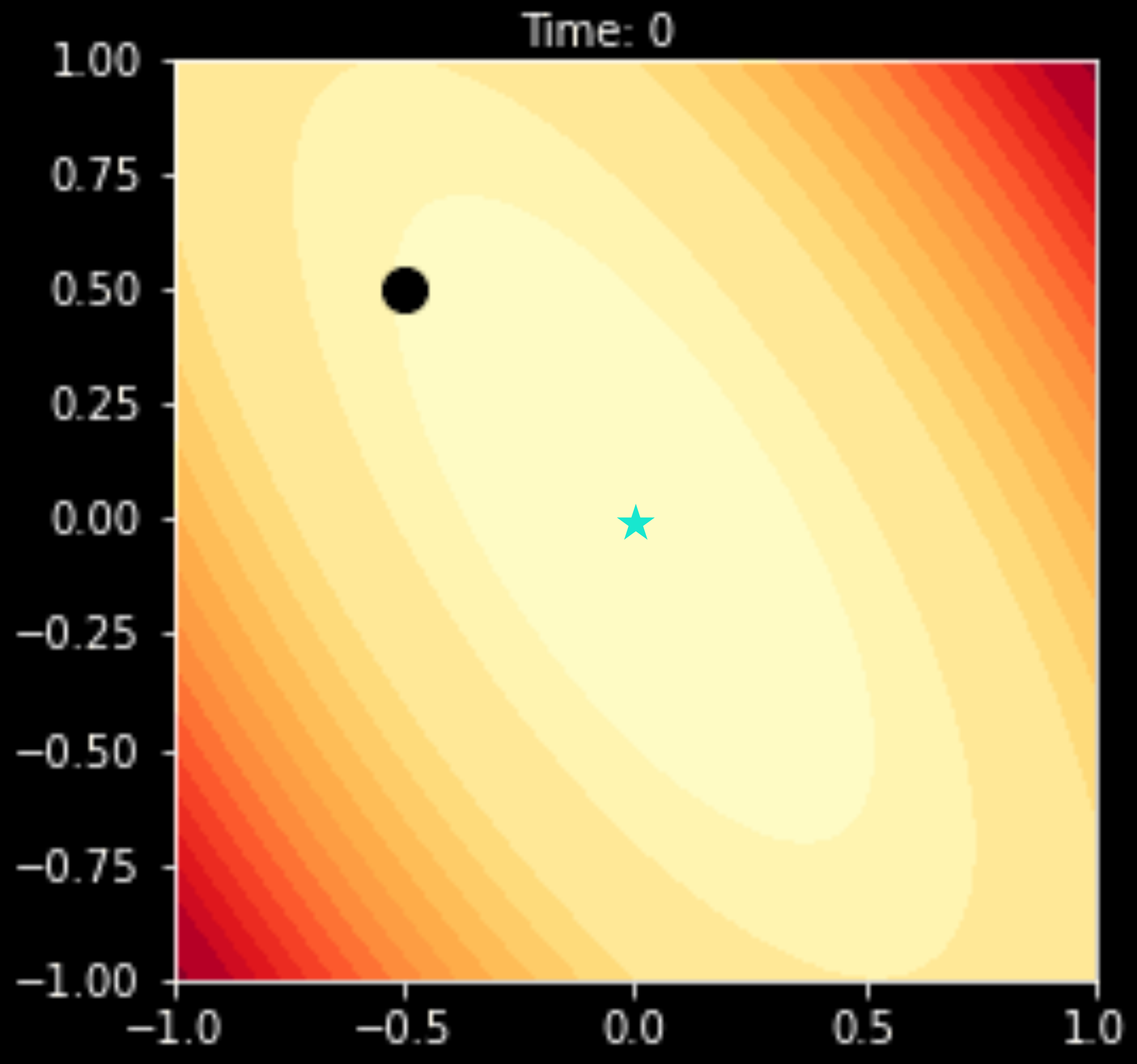
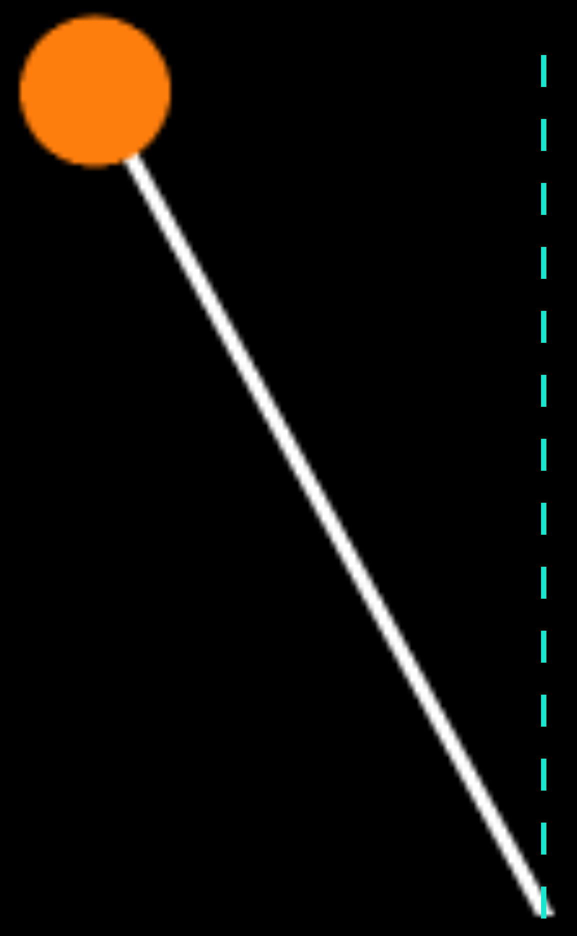
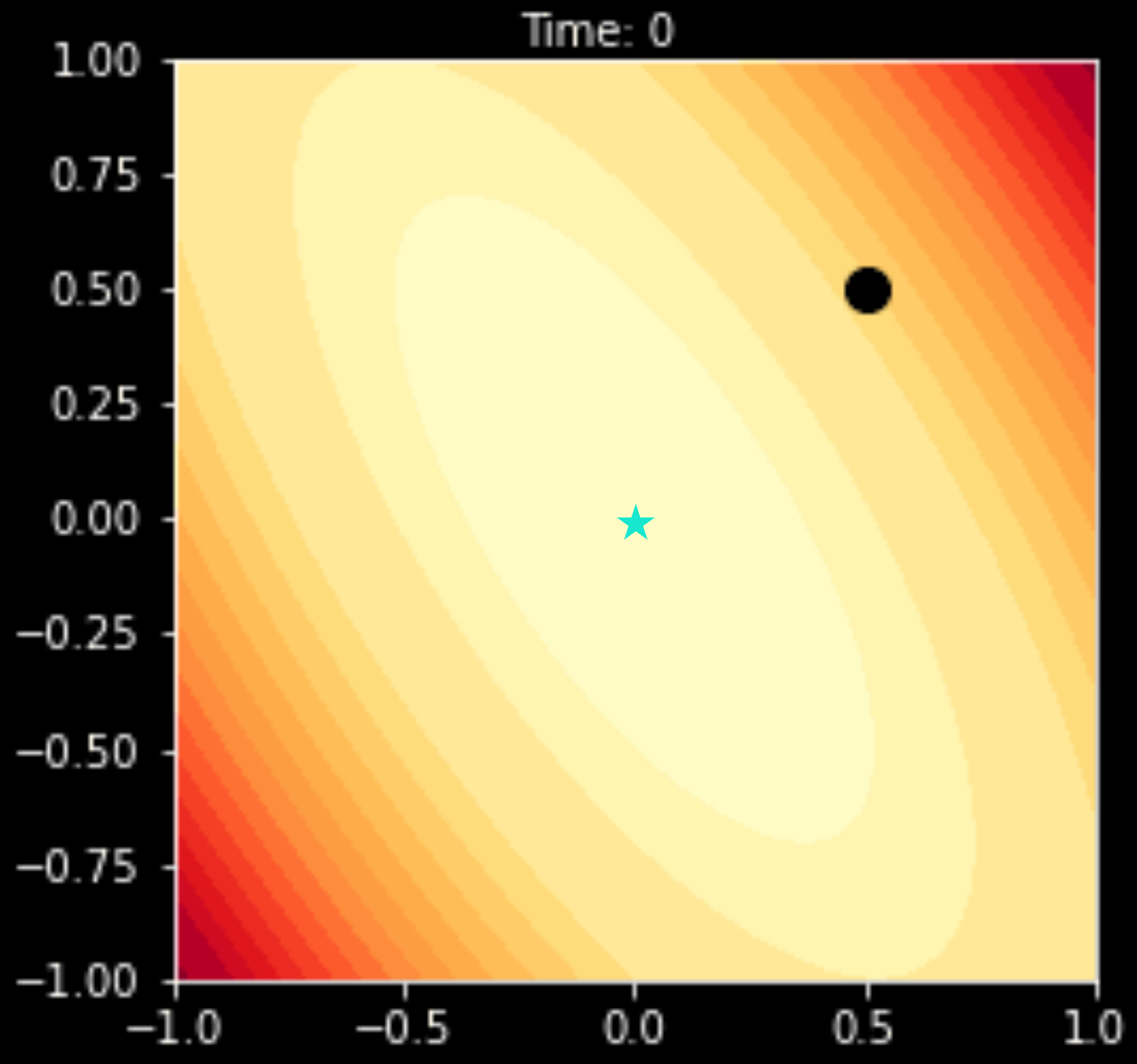




A Hard Starting Point





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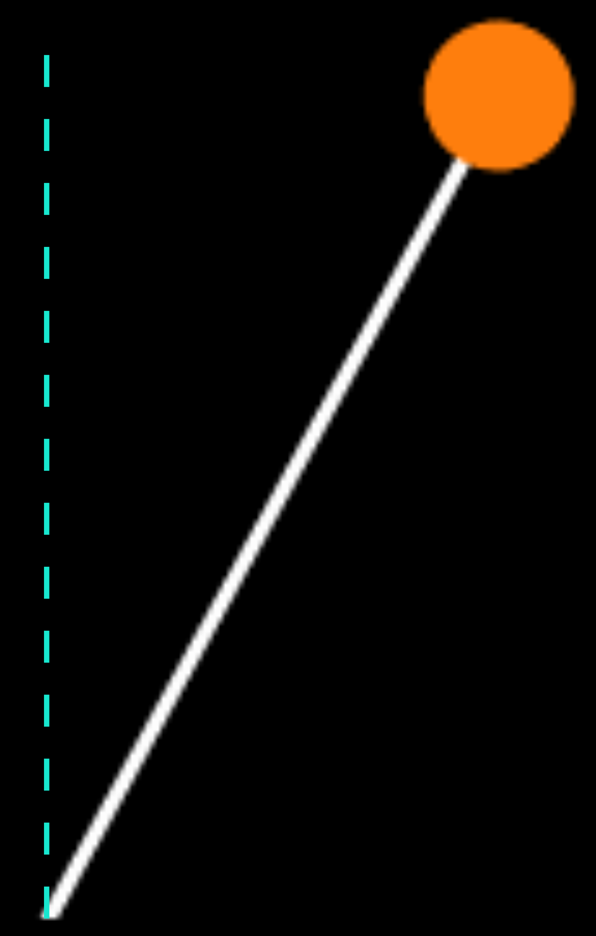

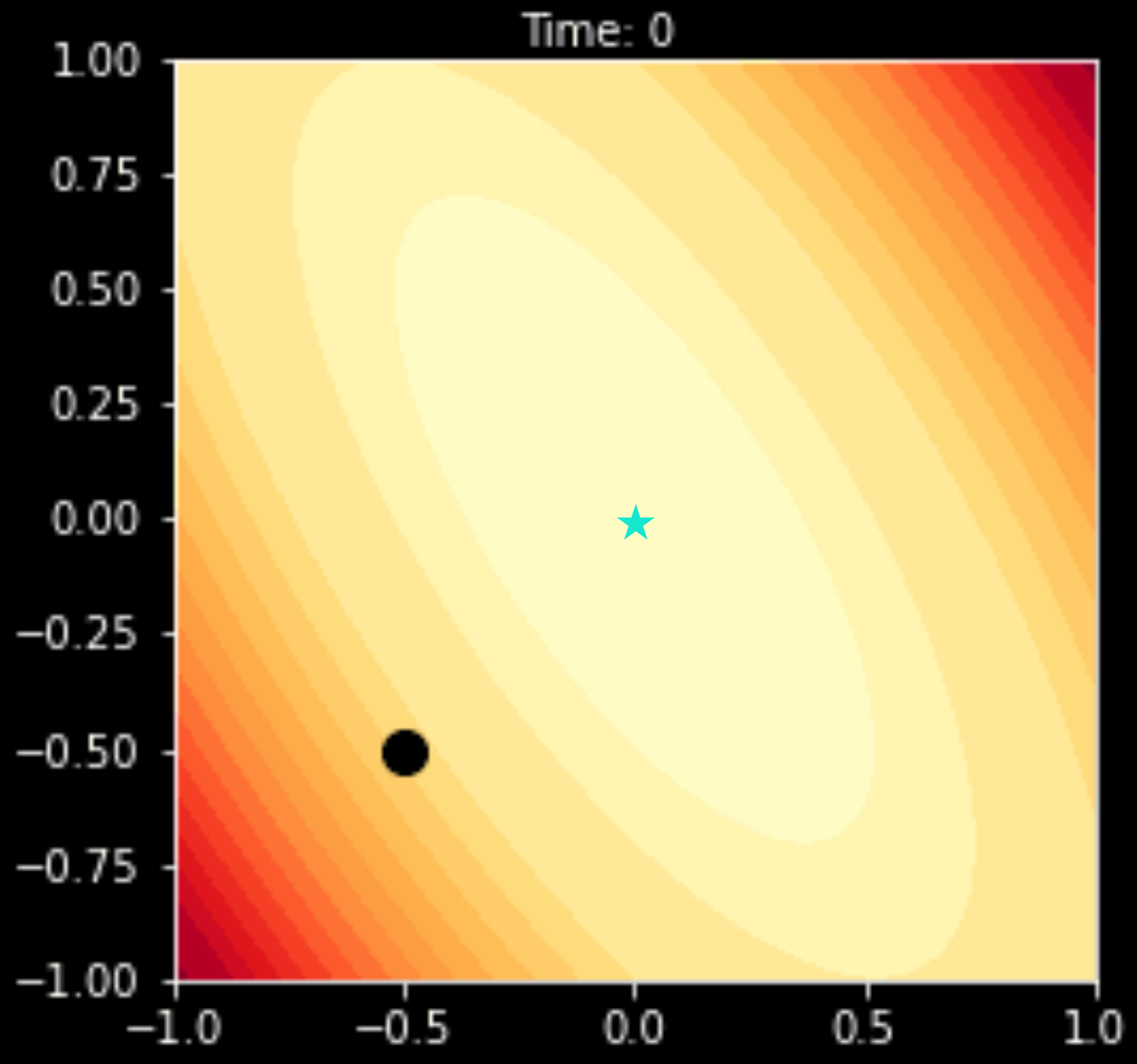

When does LQR converge?

$$V = Q + K^T R K + (A + BK)^T V (A + BK)$$

$$K = (R + B^T V B)^{-1} B^T V A$$

When the closed loop system is stable, i.e.
Eigen values of (A+BK) are inside the unit circle on the complex plane

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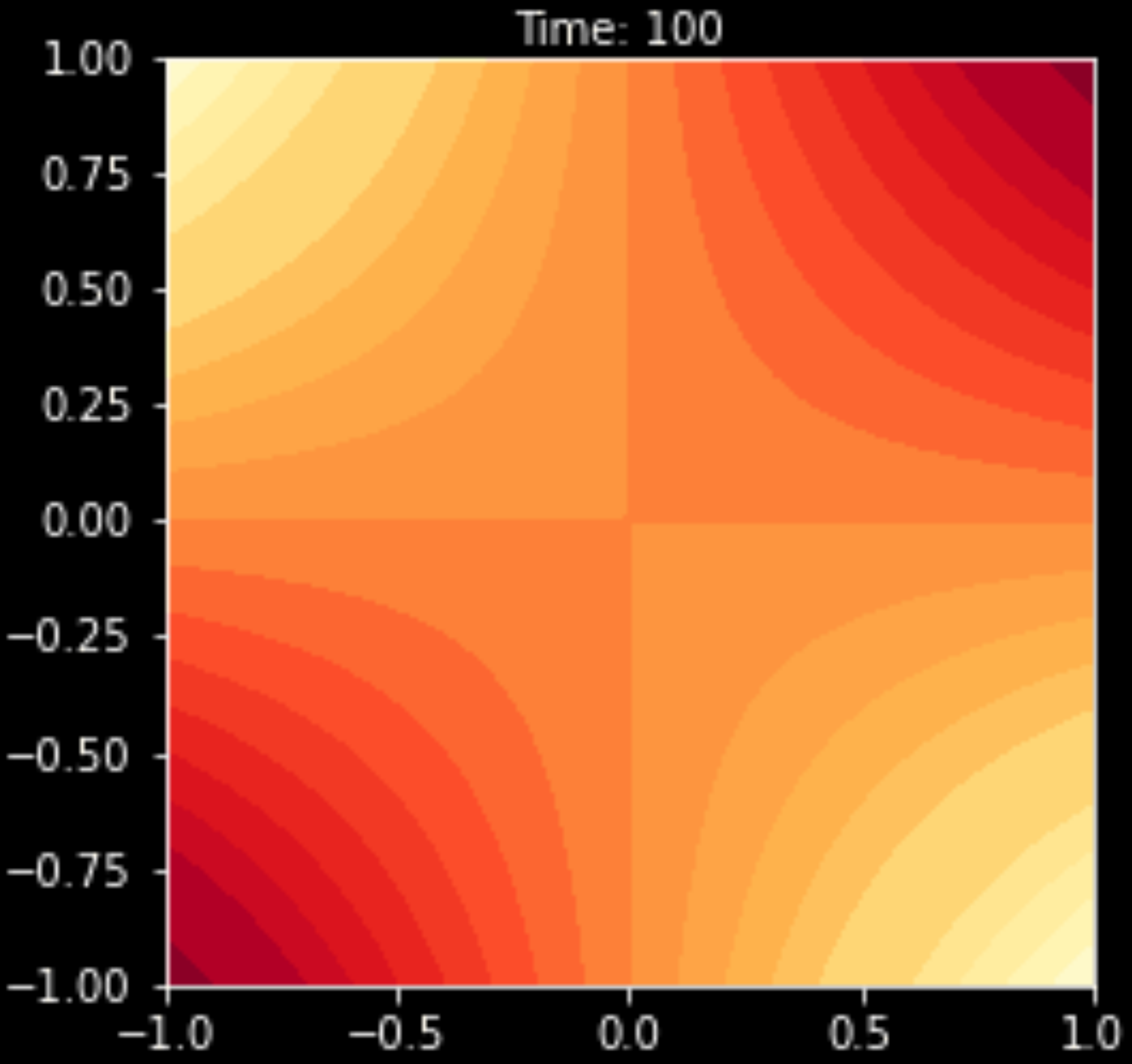
How can we find the fixed point of this equation?

Discrete time algebraic ricatti equation (DARE)

https://en.wikipedia.org/wiki/Algebraic_Riccati_equation

What if Q is not PSD?



$$x^T Q x \not\geq 0$$

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

What if R is not positive definite?

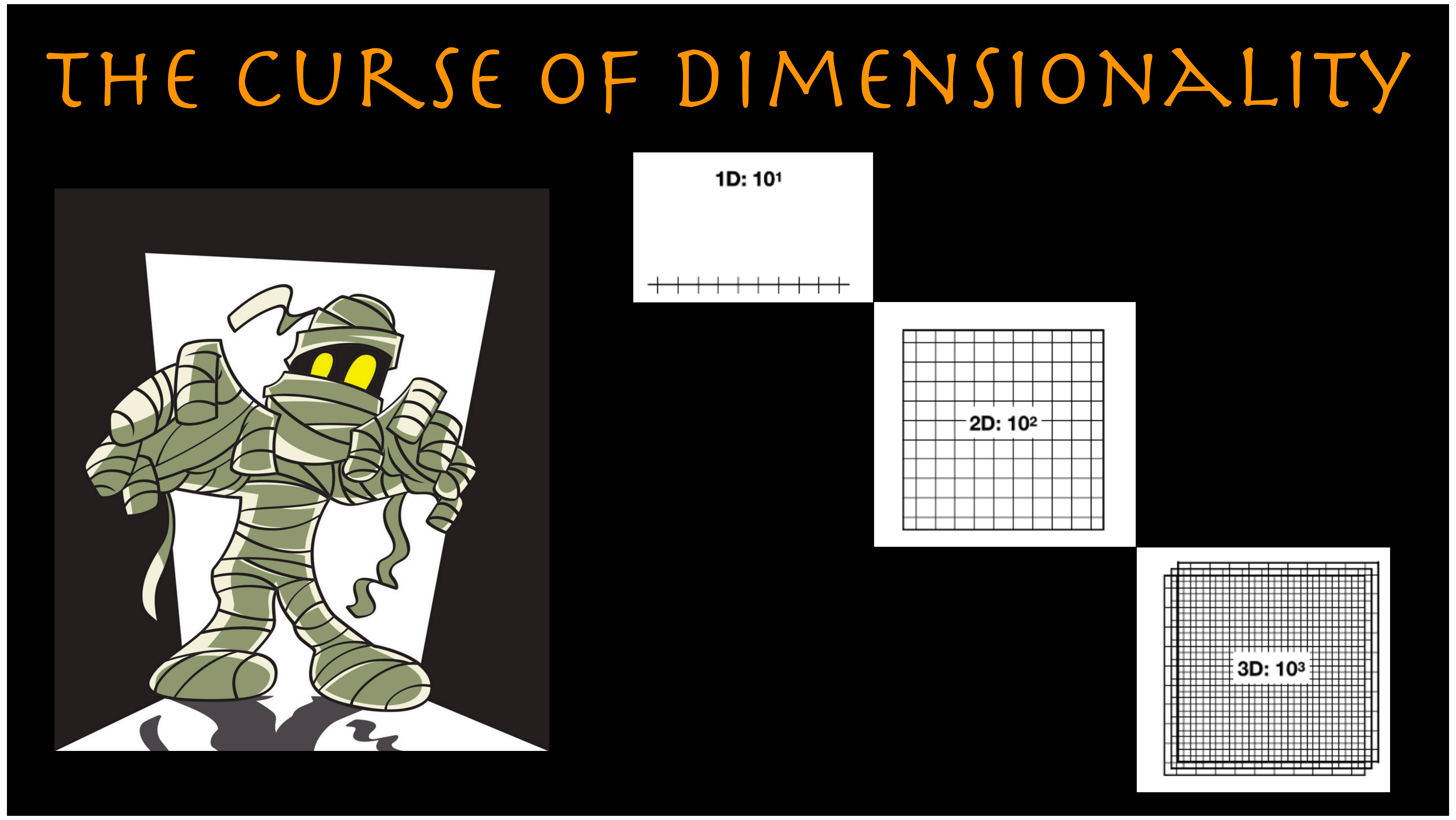
$$u^T R u \not> 0$$

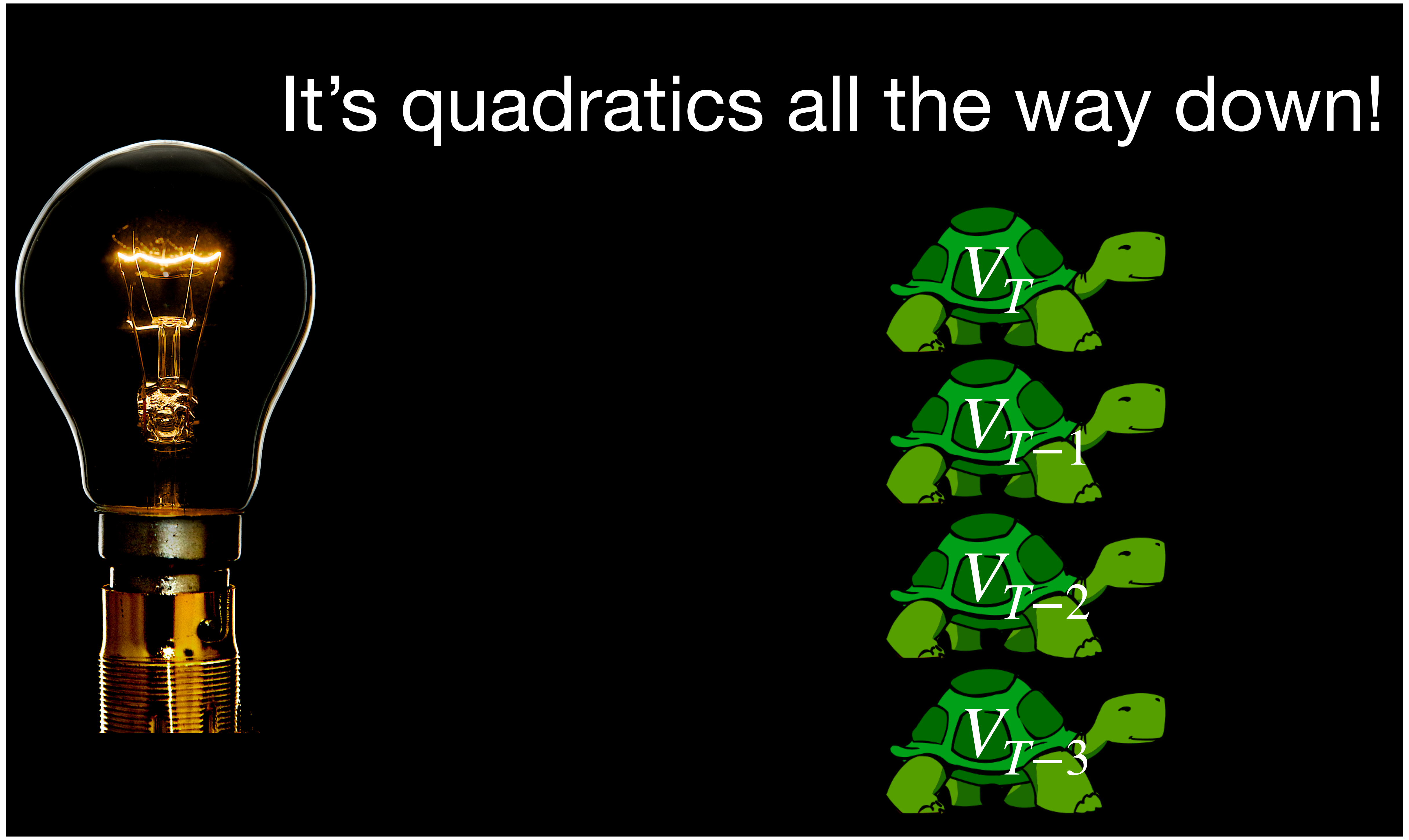
$$R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

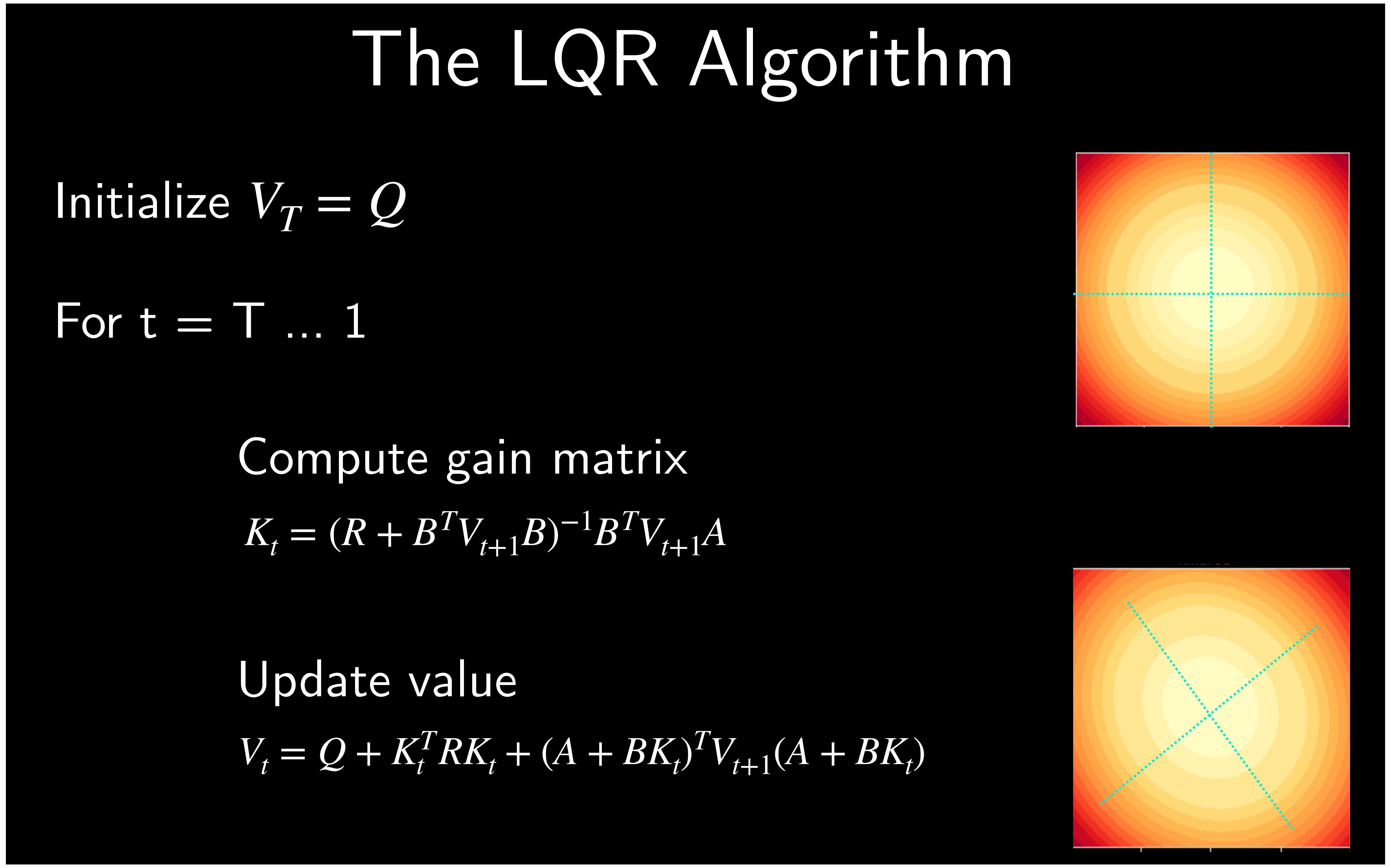
Hint: Gain matrix update?

$$K_t = (R + B^T V_{t+1} B)^{-1} B^T V_{t+1} A$$

tl;dr


THE CURSE OF DIMENSIONALITY
1D: 10¹
2D: 10²
3D: 10³


It's quadratics all the way down!
V_T
V_{T-1}
V_{T-2}
V_{T-3}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Compute gain matrix
$K_t = (R + B^T V_{t+1} B)^{-1} B^T V_{t+1} A$
Update value
$V_t = Q + K_t^T R K_t + (A + BK_t)^T V_{t+1} (A + BK_t)$