Markov Decision Process

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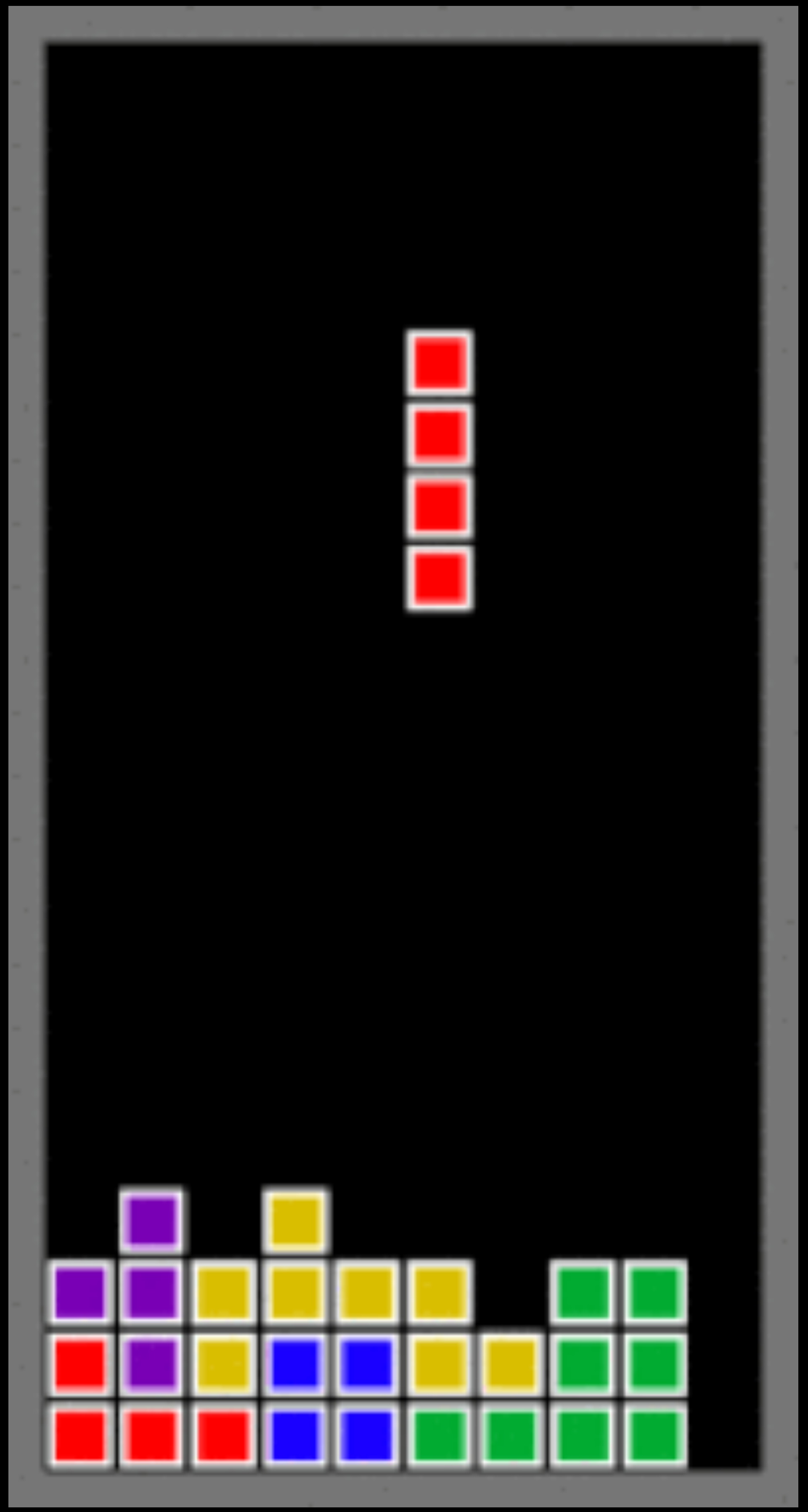
Learning

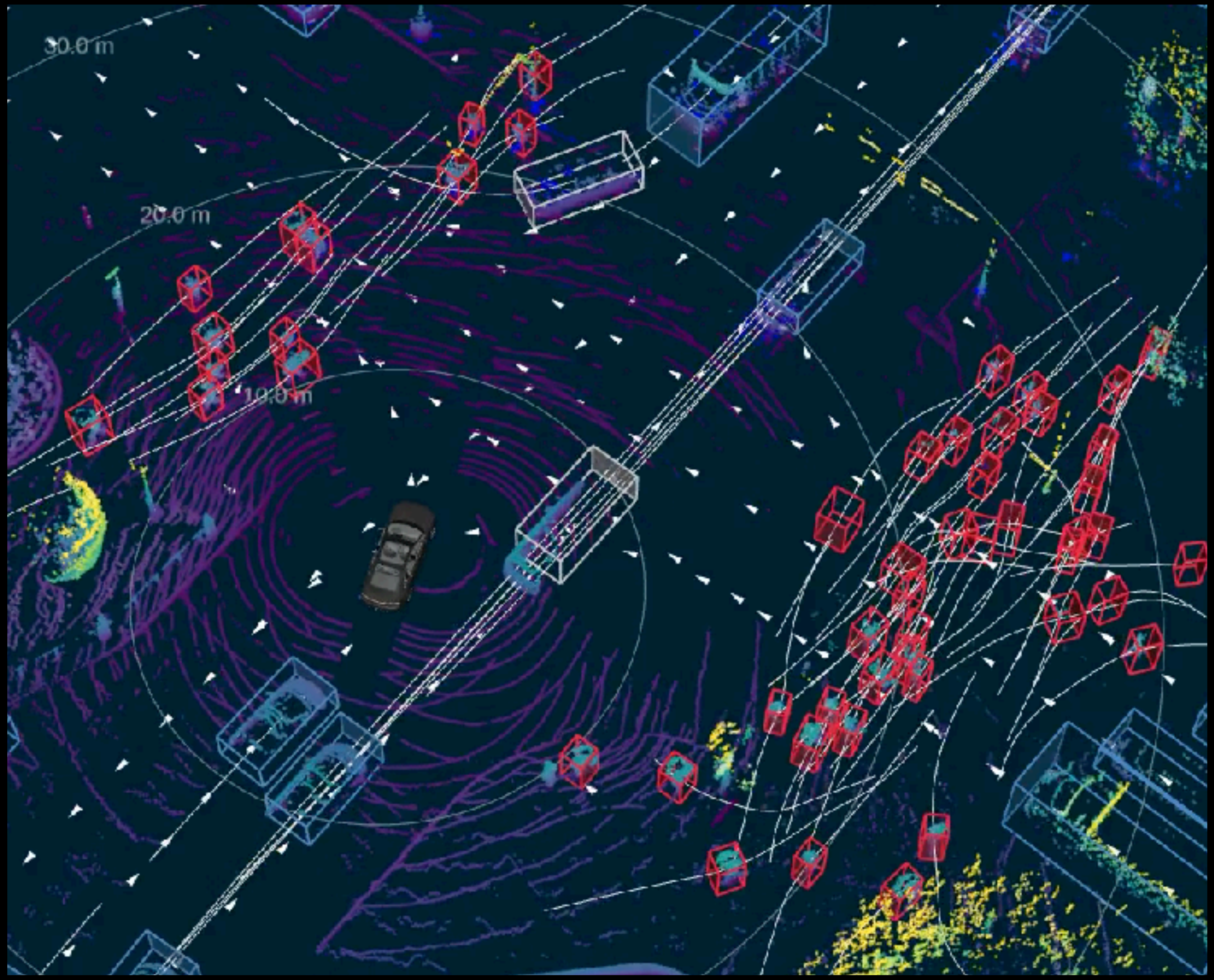
Robot
Decision
Making

Today!

Decision making across domains



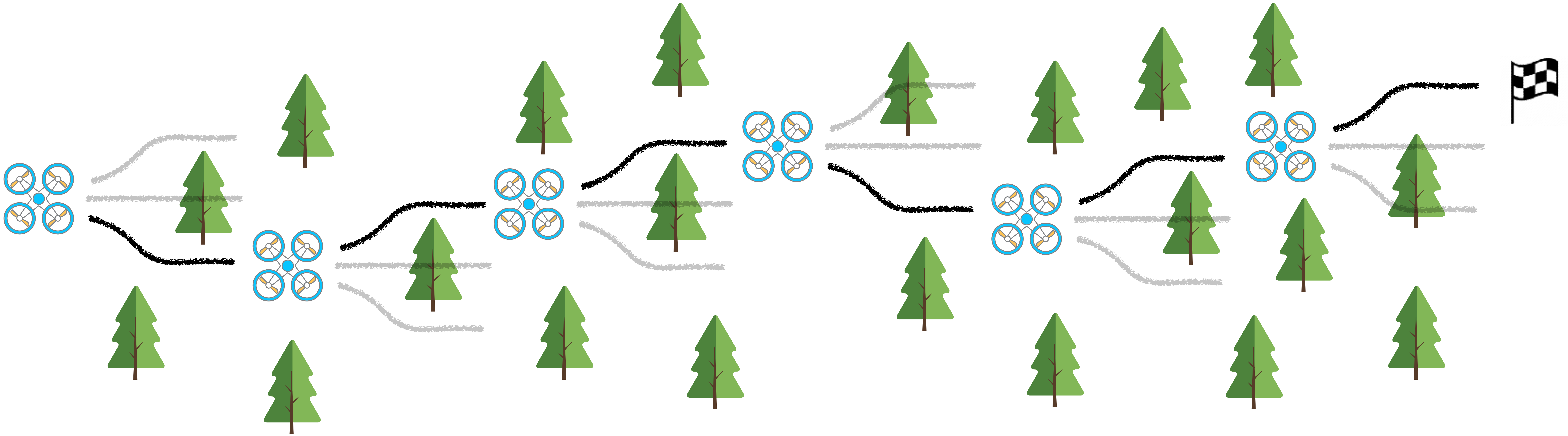
Tetris



Self-driving

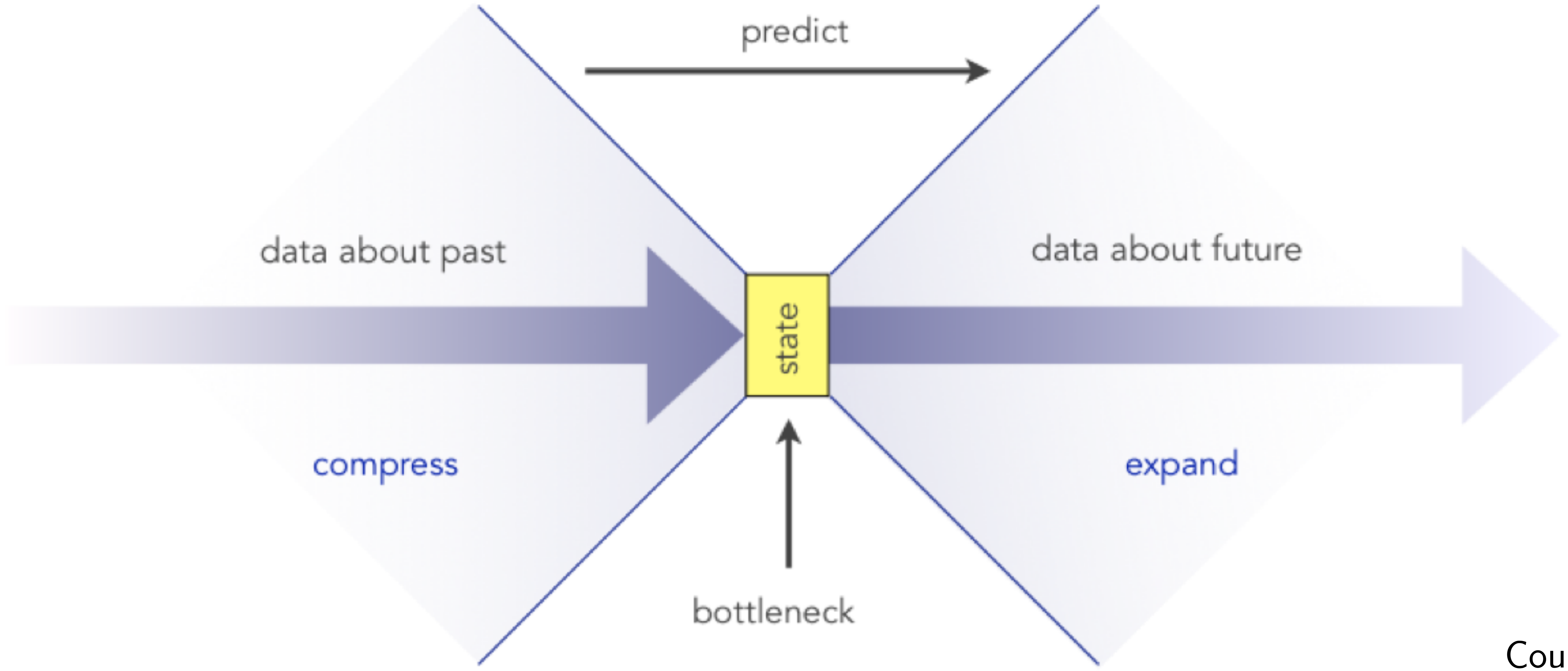
Robot Baristas

What makes decision making hard?



How do we tractably reason over a sequence of decisions?

Markov to the rescue!



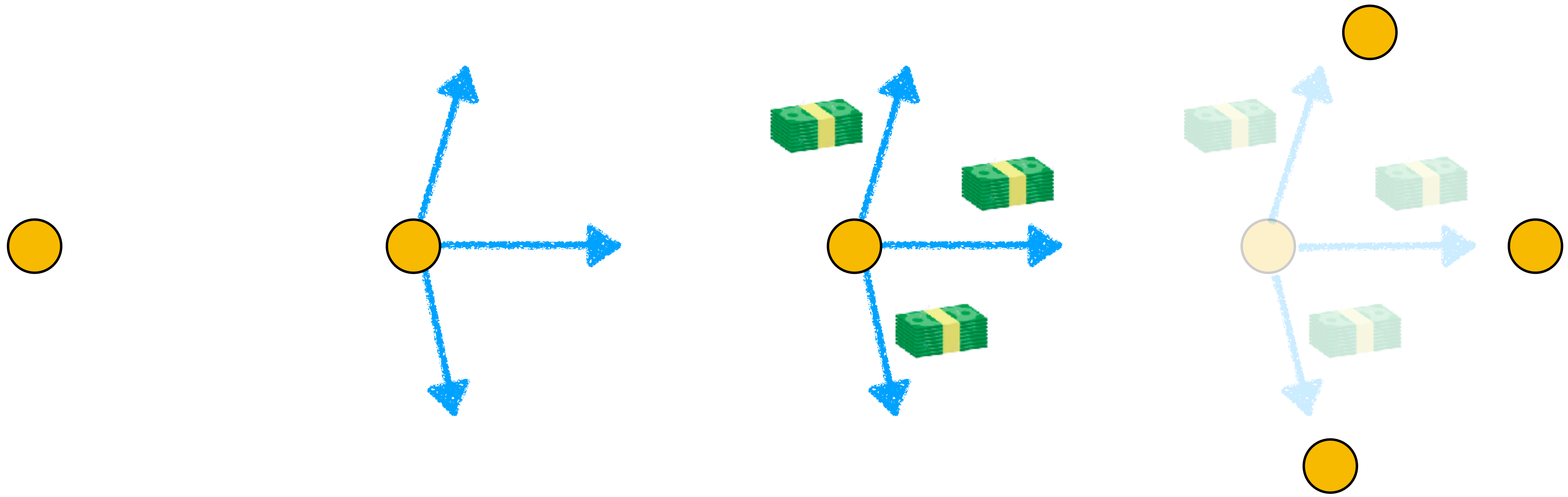
Courtesy: Byron Boots

State: statistic of history sufficient to predict the future

Markov Decision Process

A mathematical framework for modeling sequential decision making

$$< S , A , C , \mathcal{T} >$$

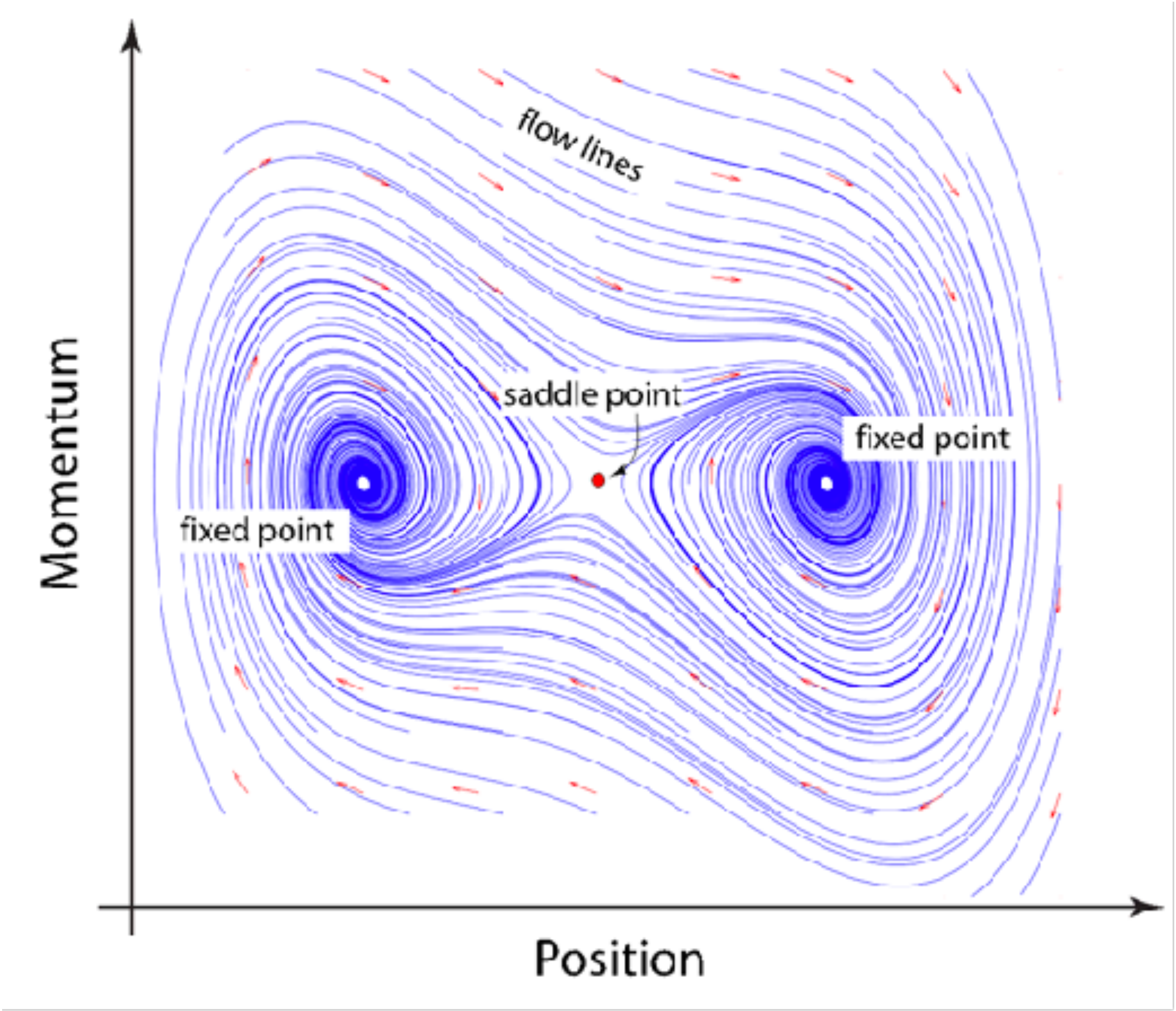


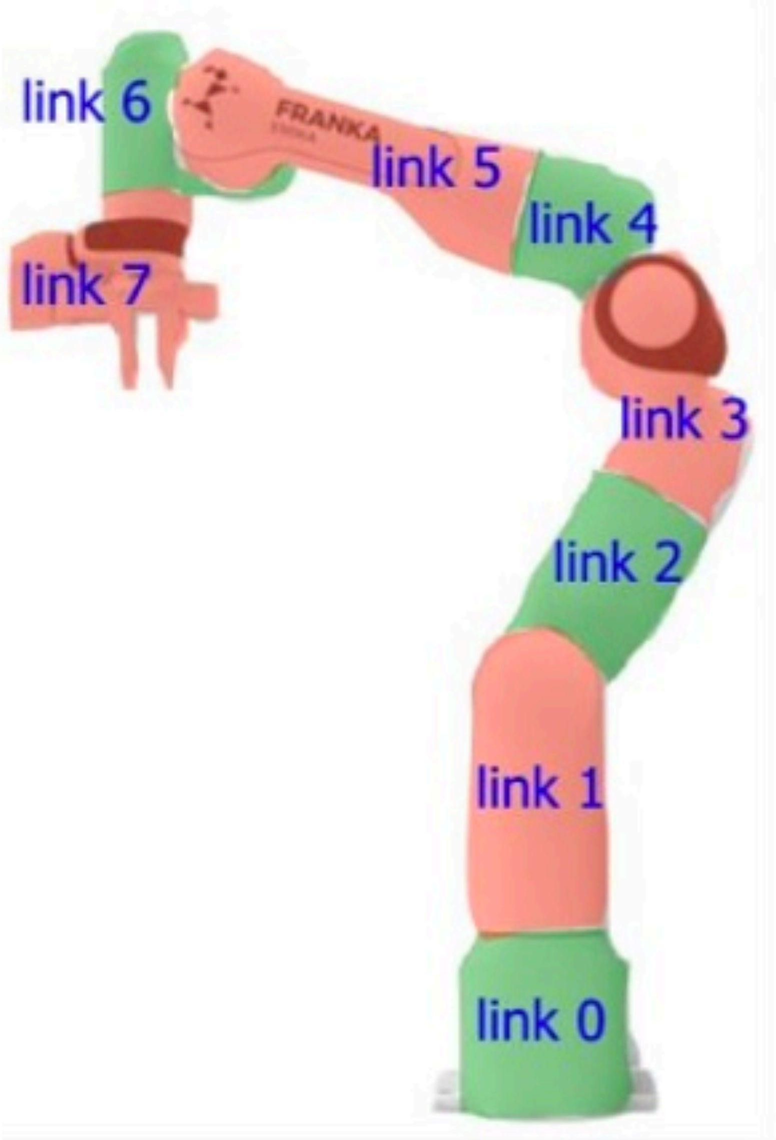
State

$< S , A , C , \mathcal{T} >$

Sufficient statistic of the system to predict future disregarding the past

- $s \in S$





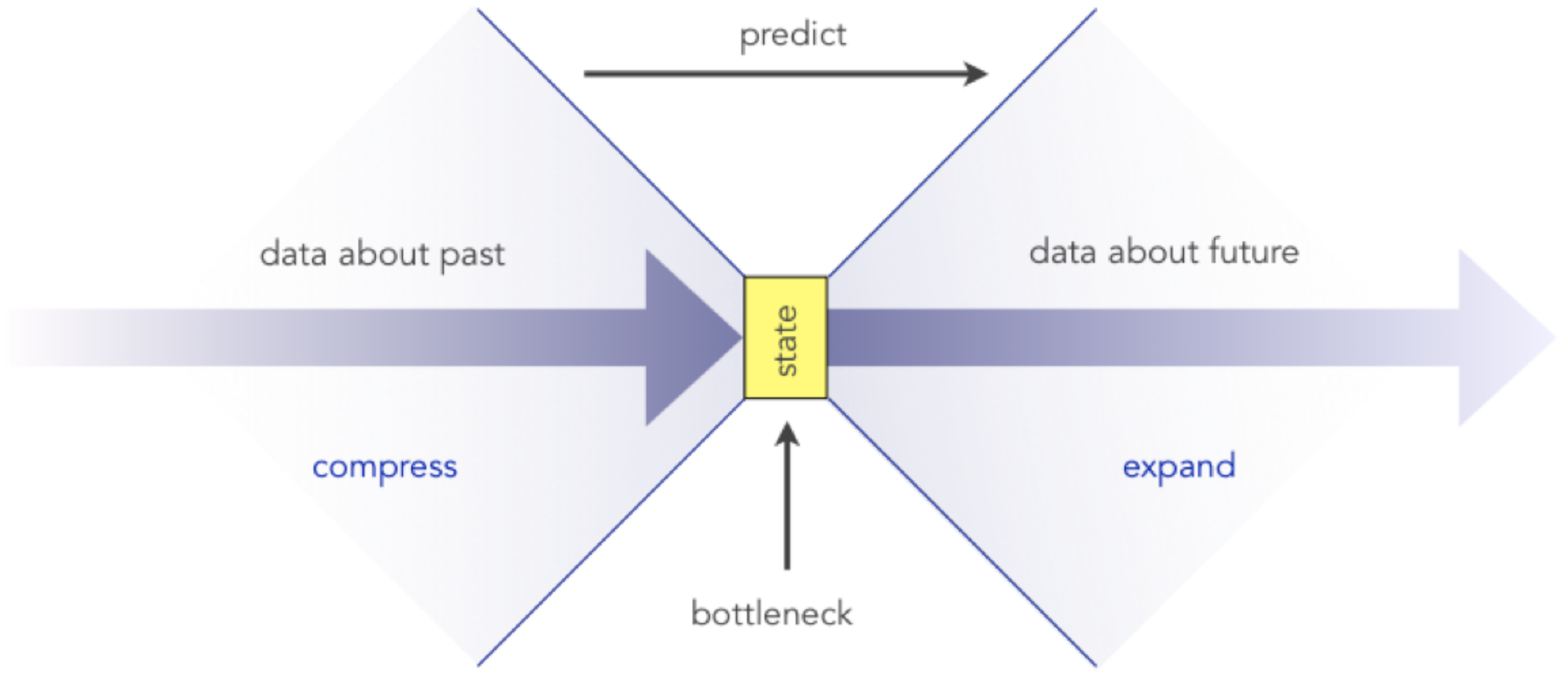
Activity!

Think-Pair-Share

Think (15 sec): Example of MDPs with shallow state?
(Current observation good enough)
Example of MDPs with deep state?

Pair: Find a partner

Share (30 sec): Partners exchange ideas

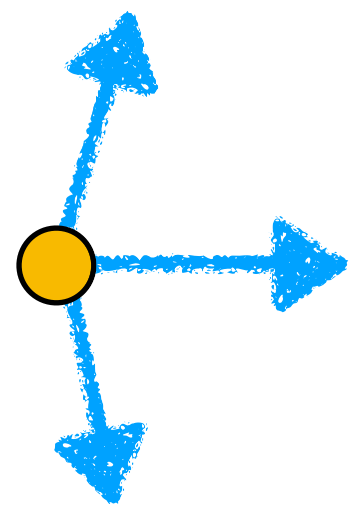


State: statistic of history sufficient to predict the future

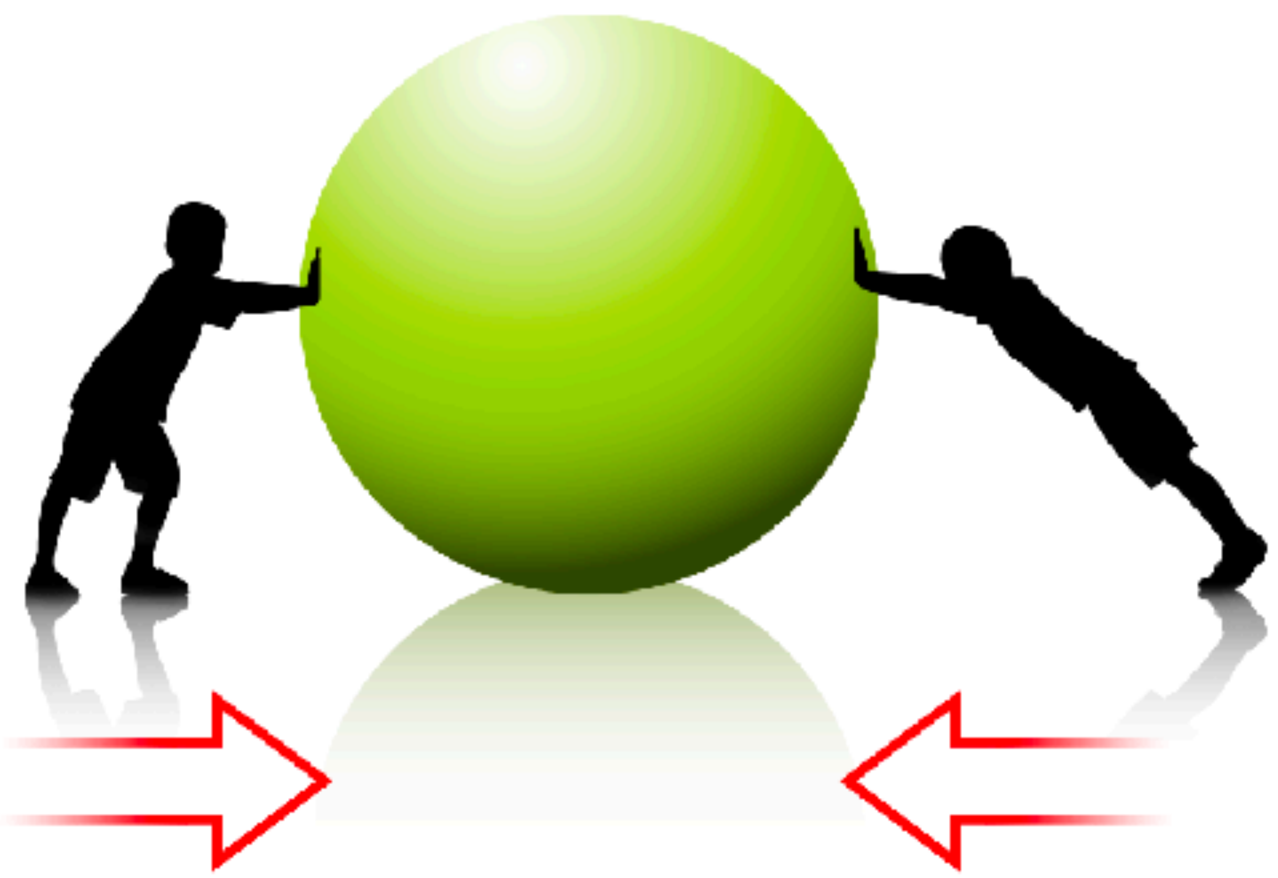
Action

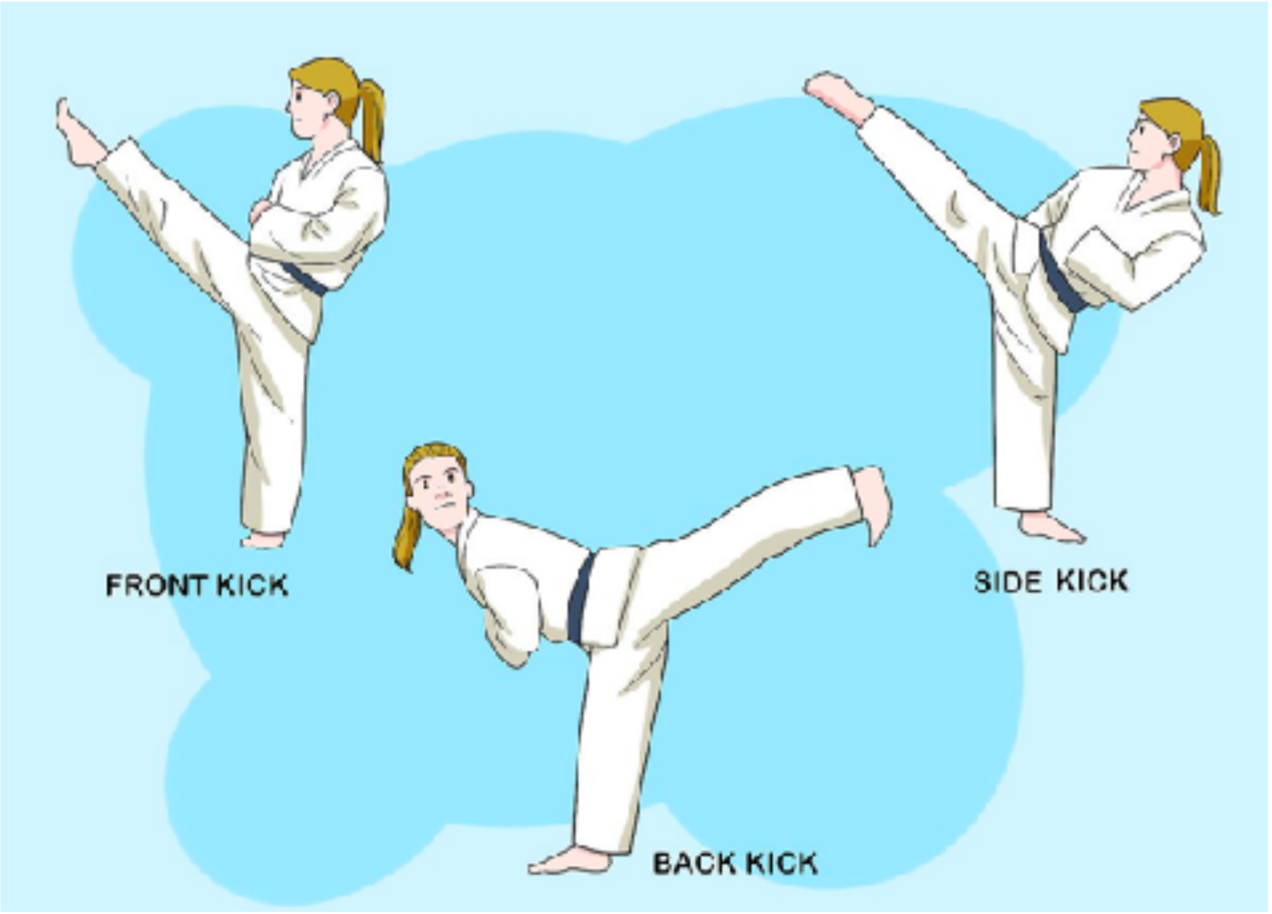
$< S , A , C , \mathcal{T} >$

Doing something:
Control action / decisions



$$a \in A$$



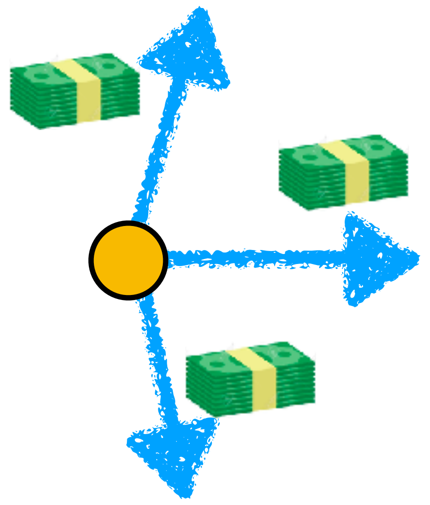


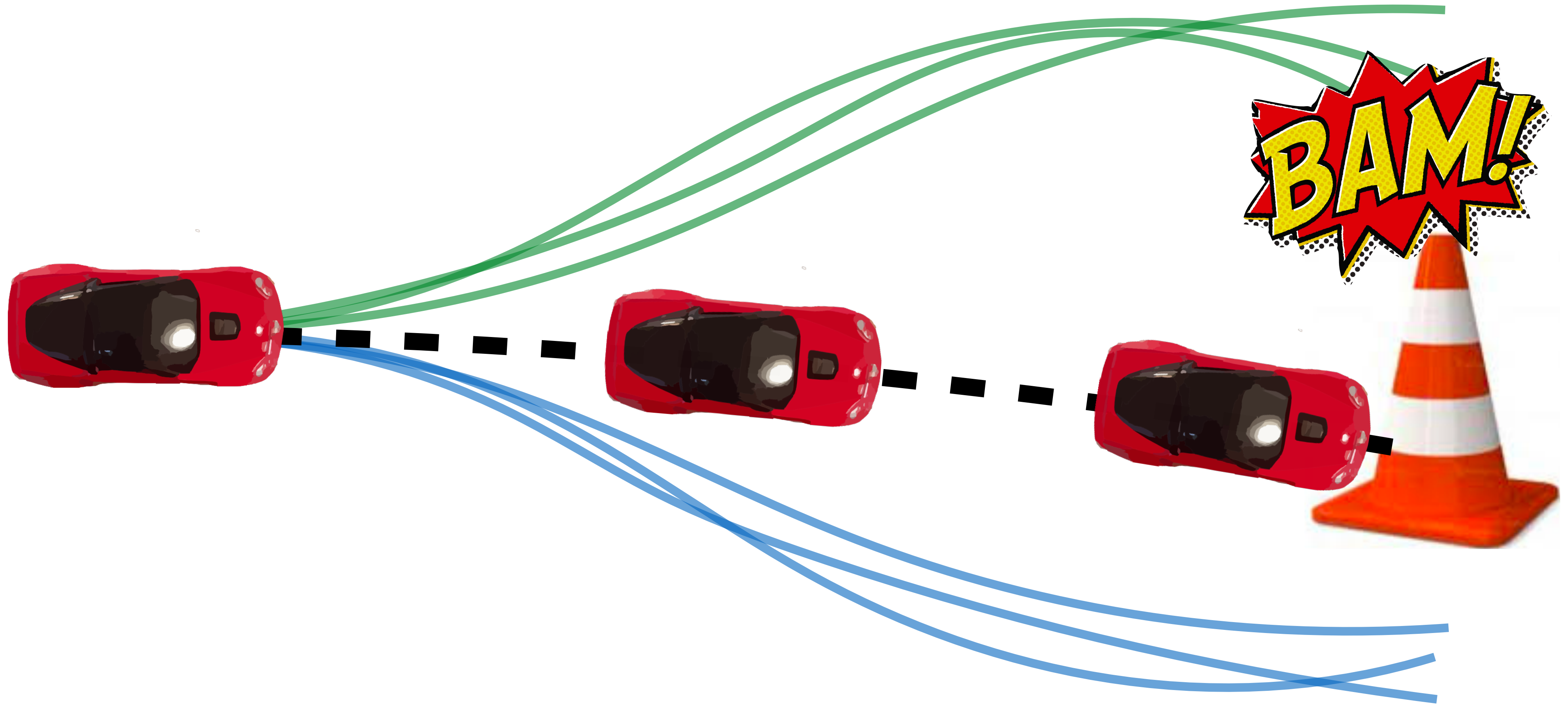
Cost

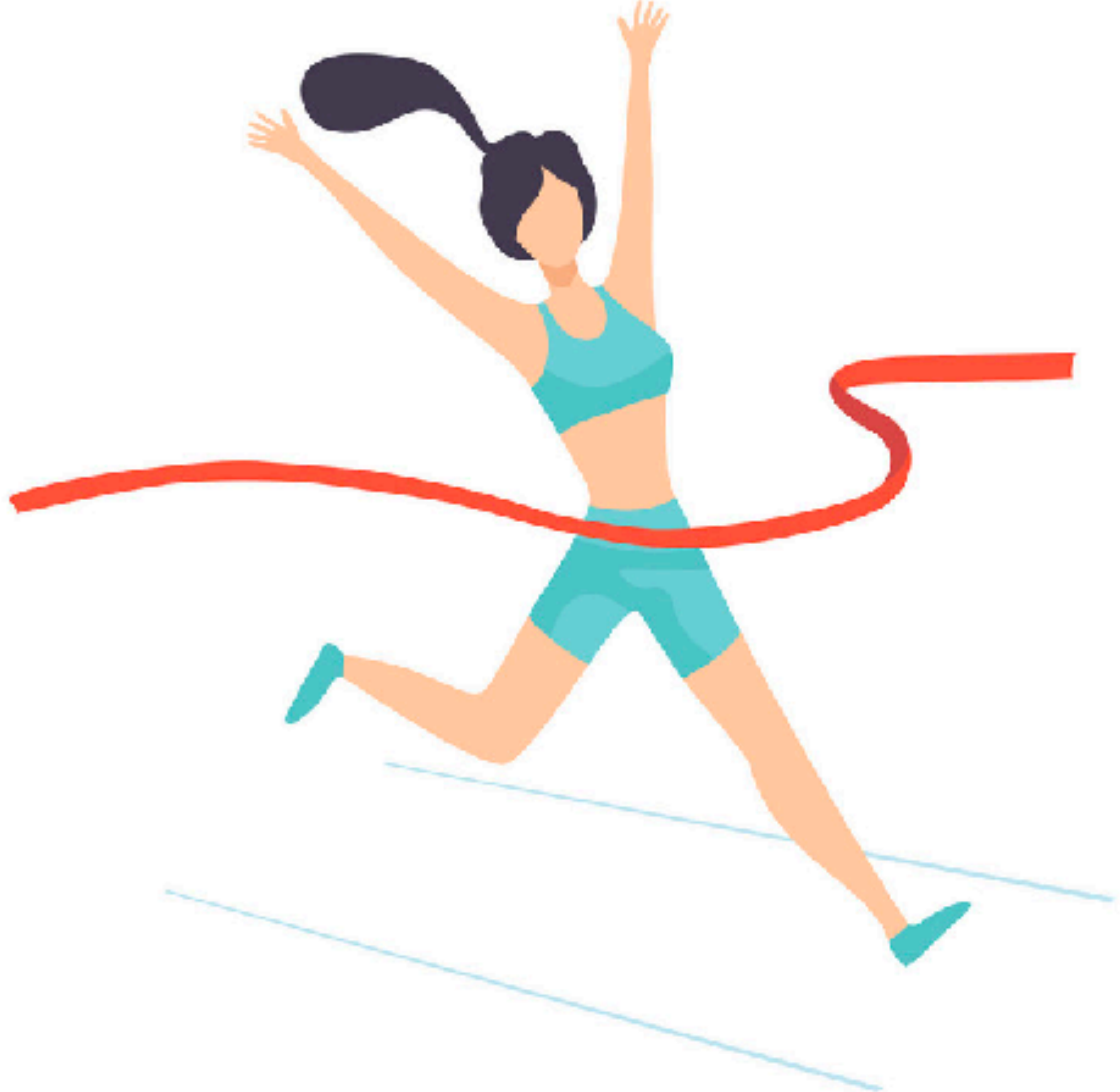
$< S , A , C , \mathcal{T} >$

The instantaneous cost of taking an action in a state

$$c(s, a)$$

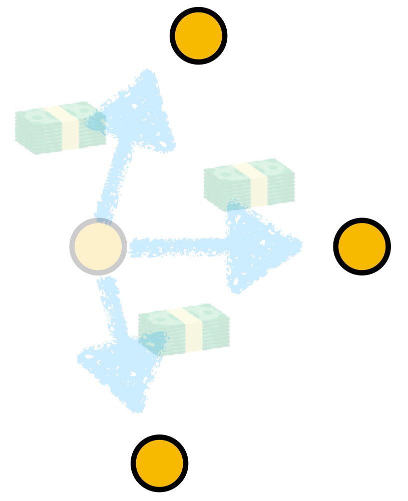






Transition

$< S , A , C , \mathcal{T} >$



The next state given state and action

$$s' = \mathcal{T}(s, a) \qquad\qquad s' \sim \mathcal{T}(s, a)$$

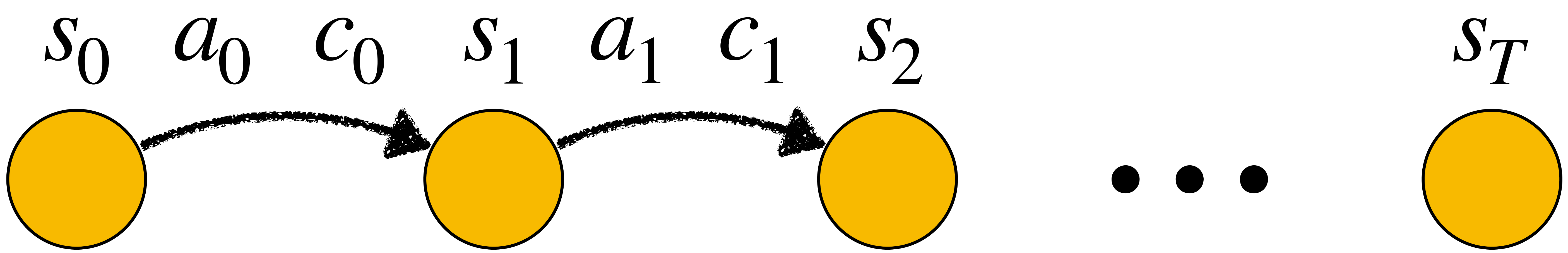
Deterministic — Stochastic

Poll!

Markov Decision ~~Process~~ → Problem

Includes things to define an optimization problem

Horizon $T \in \mathbb{N}$



Discount $0 \leq \gamma \leq 1$

Return: $c_0 + \gamma c_1 + \cdots \gamma^{T-1} c_{T-1}$

(Costs are more valuable if they happen soon)

Markov Decision ~~Process~~ → Problem

Policy

$$\pi \in \Pi$$

$$\pi : s_t \to a_t$$ (Deterministic)

$$\pi : s_t \to P(a_t)$$ (Stochastic)

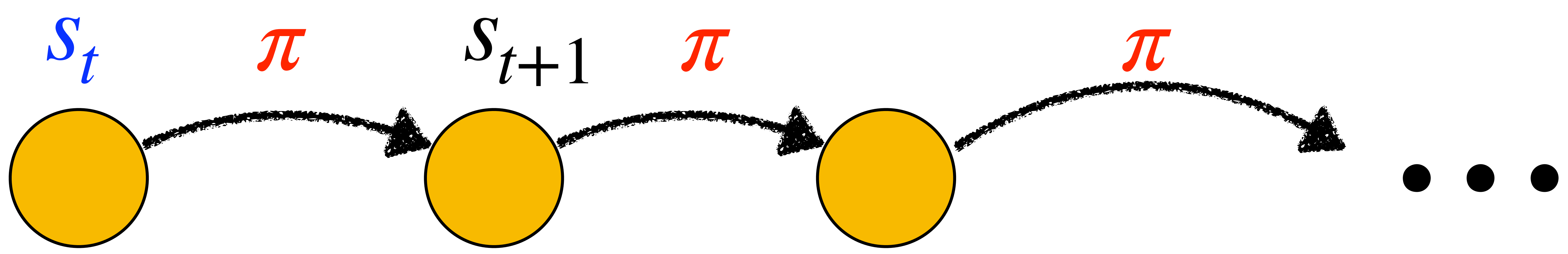
A function that maps state (and time) to action

Objective Function

$$\min_{\pi} \mathop{\mathbb{E}}_{\substack{a_t \sim \pi(s_t) \\ s_{t+1} \sim \mathcal{T}(s_t, a_t)}} [\sum_{t=0}^{T-1} \gamma^t c(s_t, a_t)]$$

Find policy that minimizes sum of discounted future costs

Value of a state

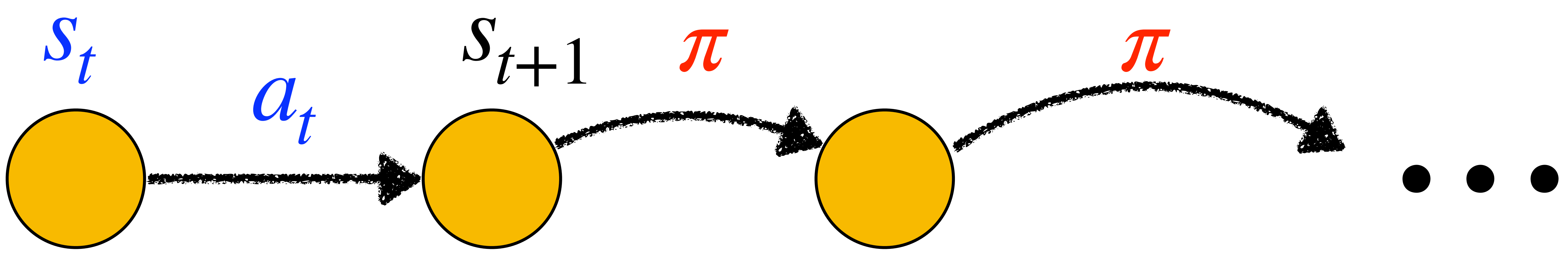


$$V^{\pi}(s_t) \quad = c_t + \quad \gamma c_{t+1} + \quad \gamma^2 c_{t+2} +$$

Expected discounted sum of cost from starting at a state
and following a policy from then on

$$\pi^* = \arg\min_{\pi} \mathbb{E}_{s_0} V^{\pi}(s_0)$$

Value of a state-action

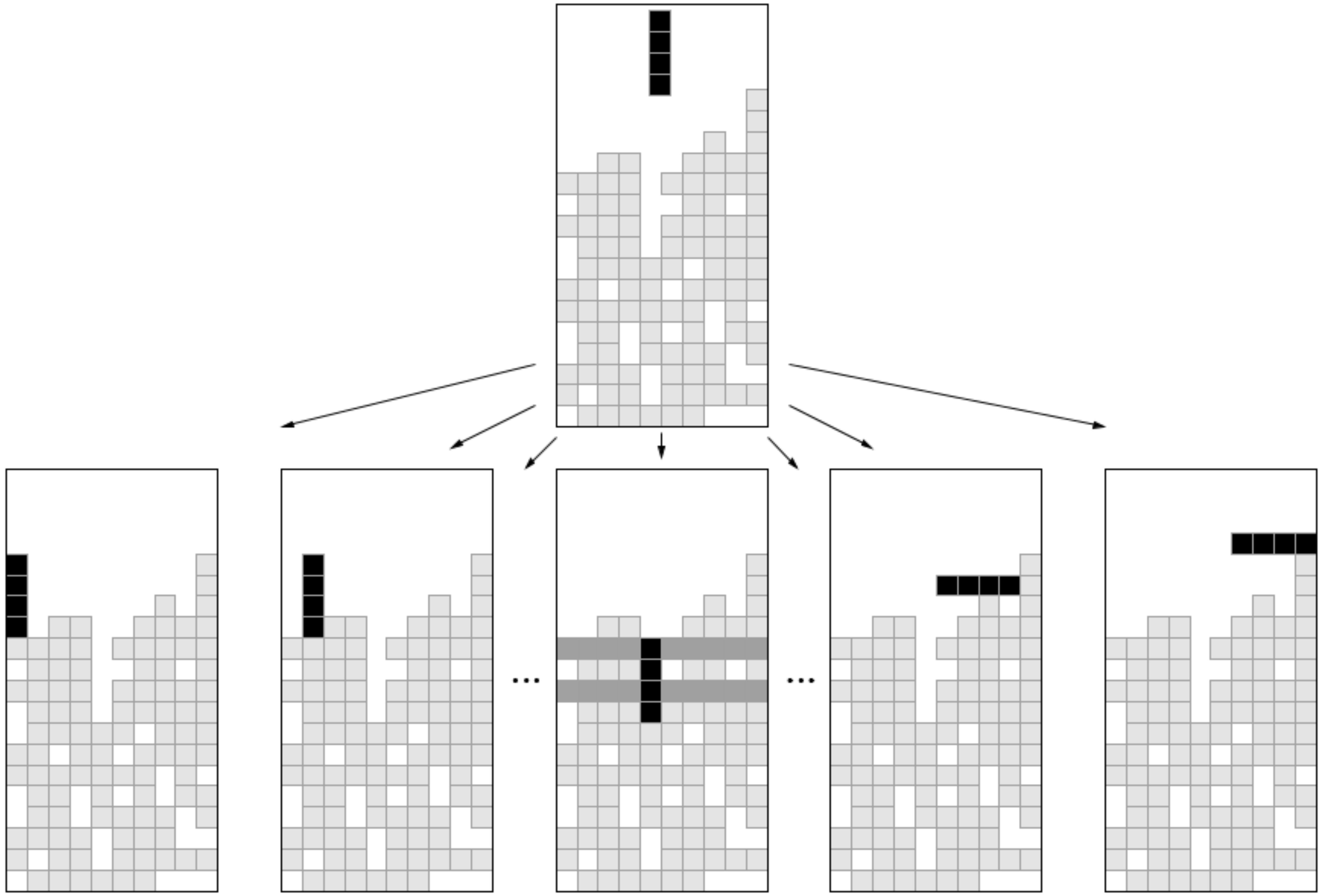


$$Q^{\pi}(s_t, a_t) = c_t + \gamma c_{t+1} + \gamma^2 c_{t+2} + \cdots$$

Expected discounted sum of cost from starting at a state, executing action and following a policy from then on

$$Q^{\pi}(s_t, a_t) = c(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1} \sim \mathcal{T}(s_t, a_t)} V^{\pi}(s_{t+1})$$

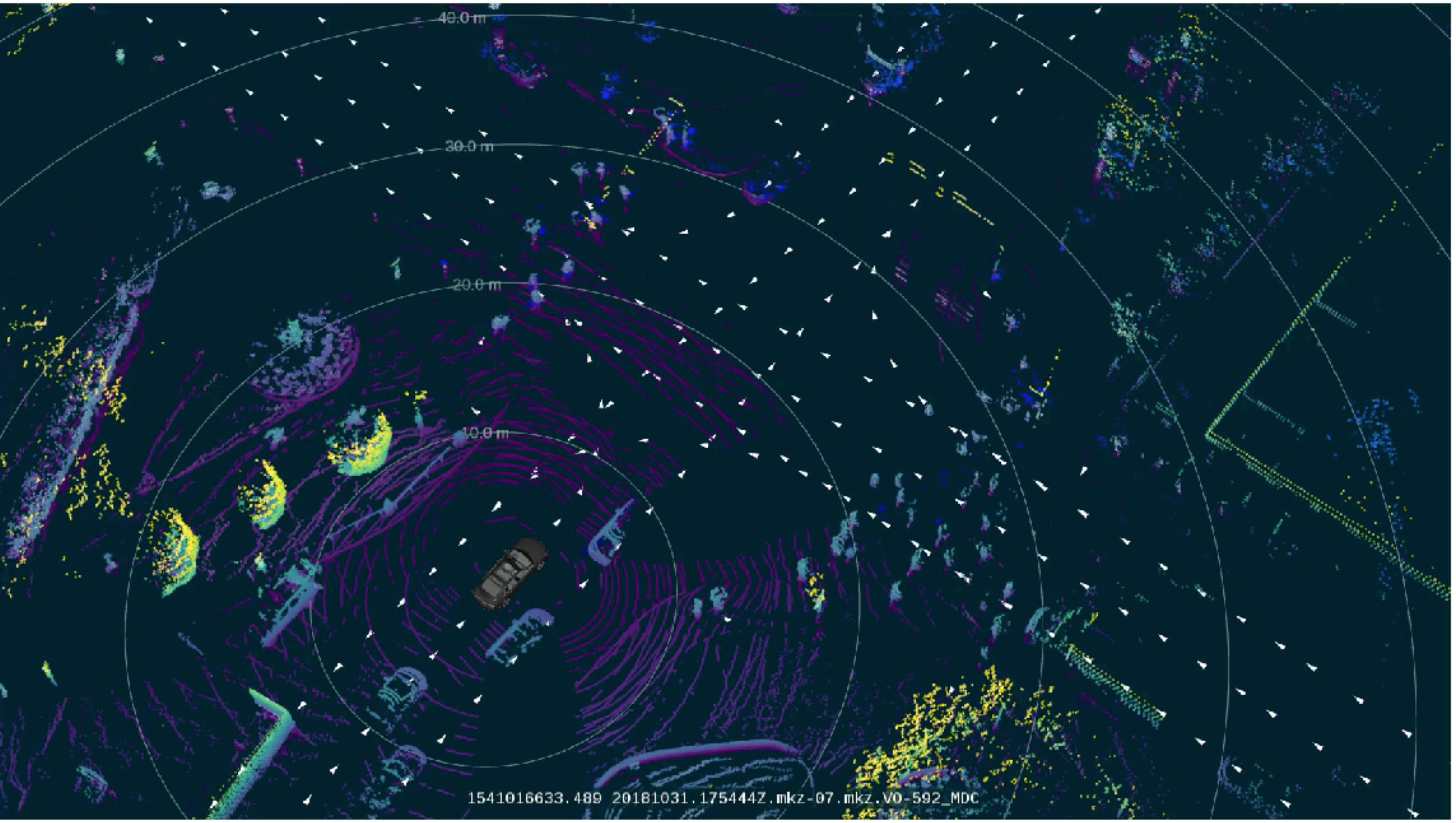
Example 1: Tetris!



$$< S , A , C , \mathcal{T} >$$

?

Example 2: Self-driving



$$< S , A , C , \mathcal{T} >$$

?

Example 3: Coffee making robot

$$< S , A , C , \mathscr{T} >$$

?

Solving MDPs

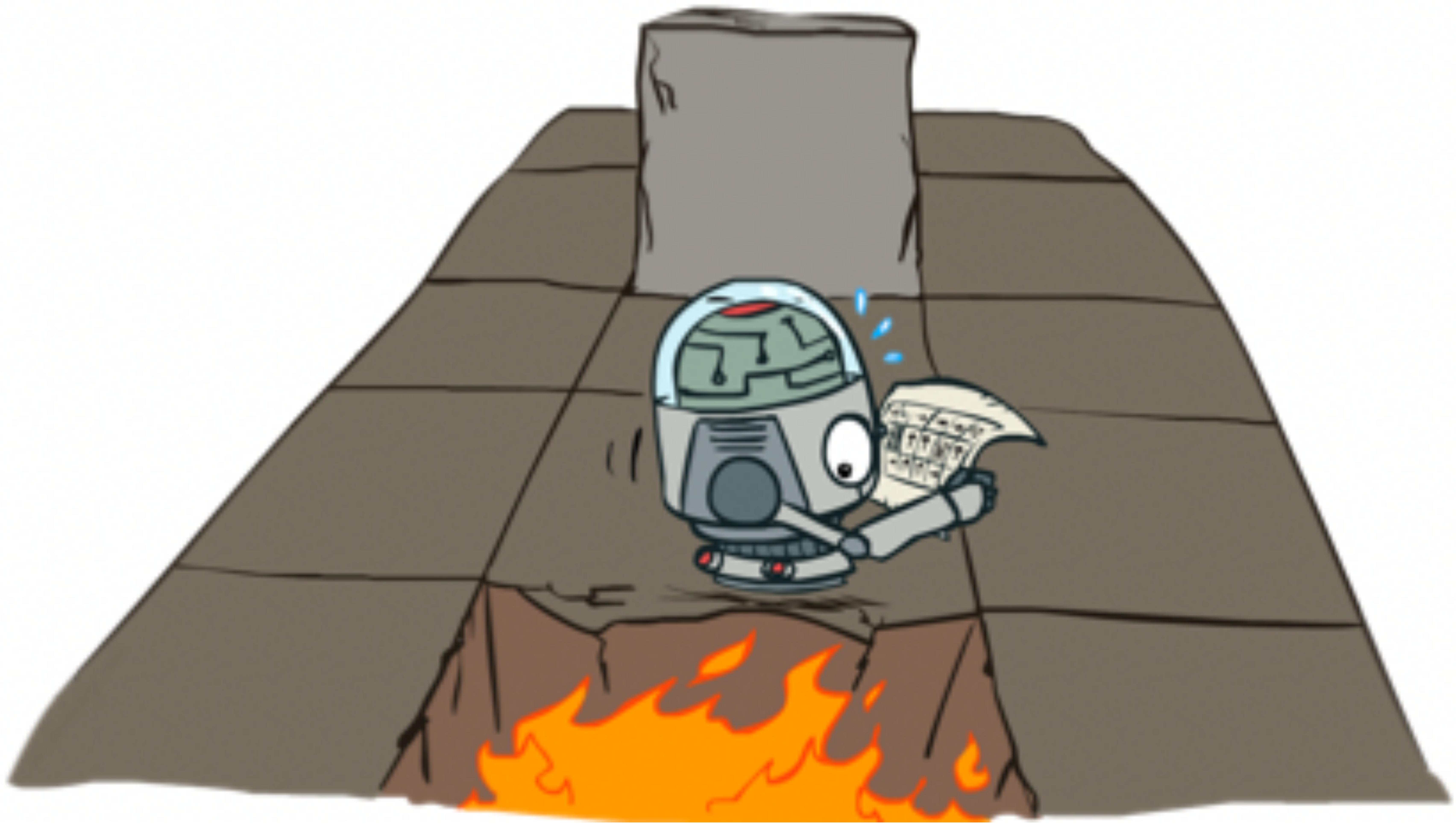
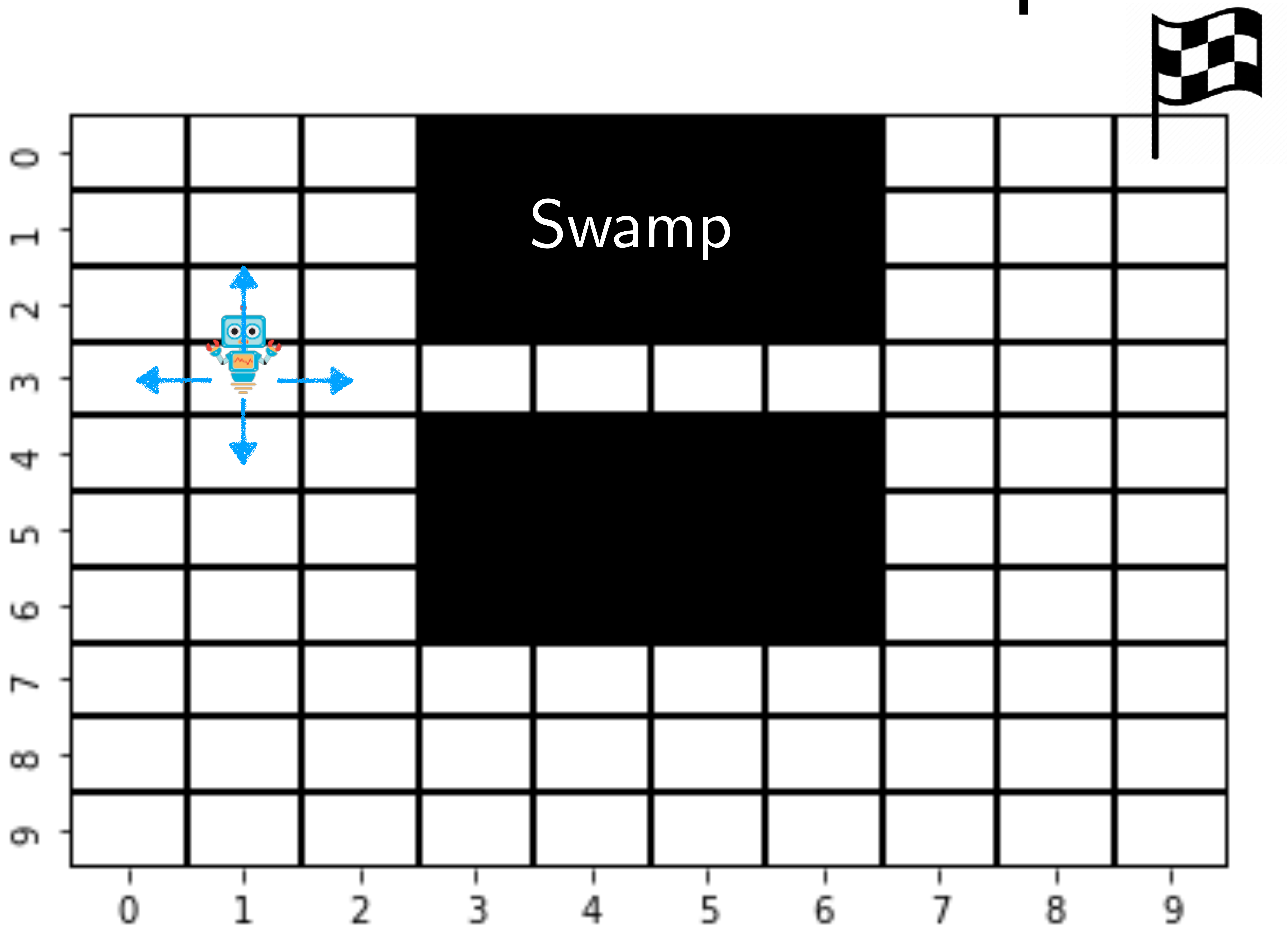


Image courtesy Dan Klein

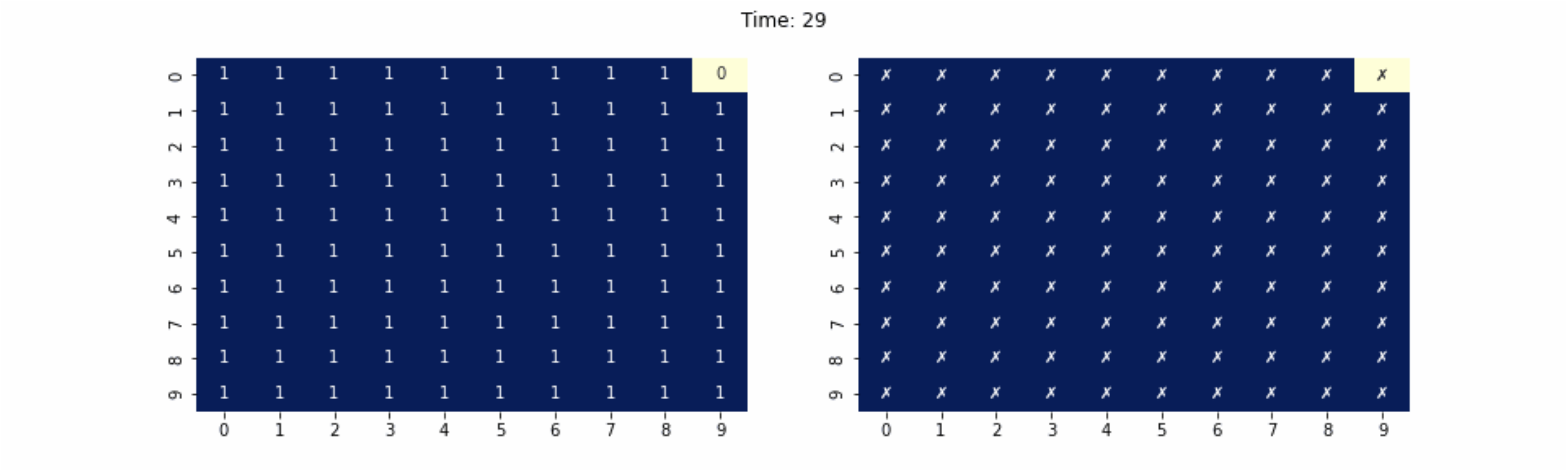
Setup



$$< S , A , C , \mathscr{T} >$$

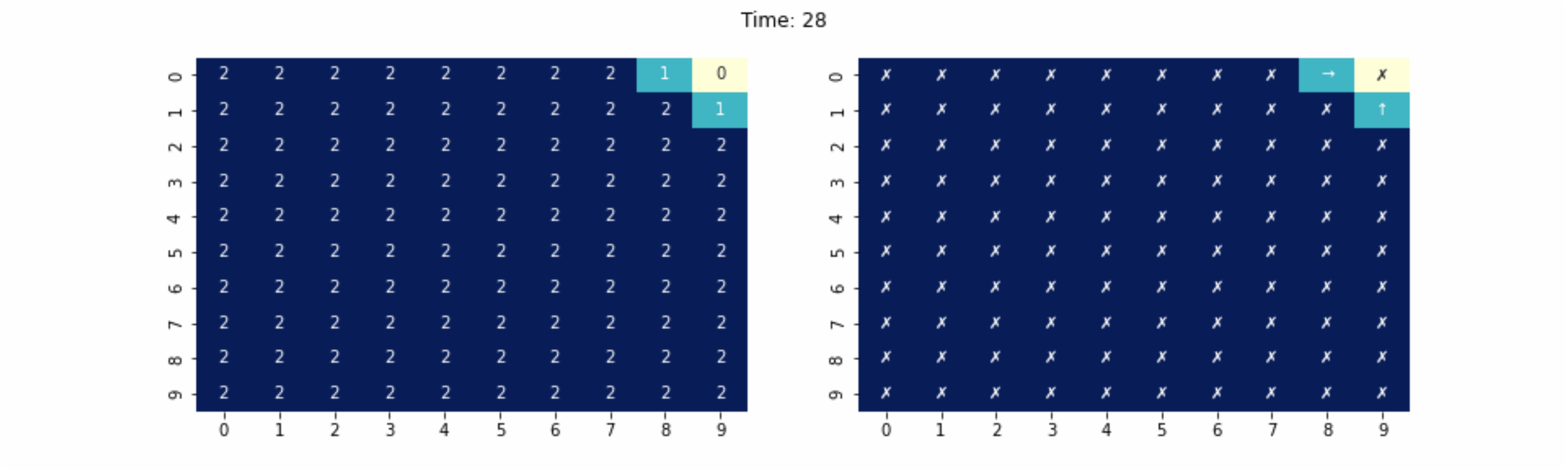
- Two absorbing states: Goal and Swamp
- Cost of each state is 1 till you reach the goal
- Let's set T = 30

What is the optimal value at T-1?



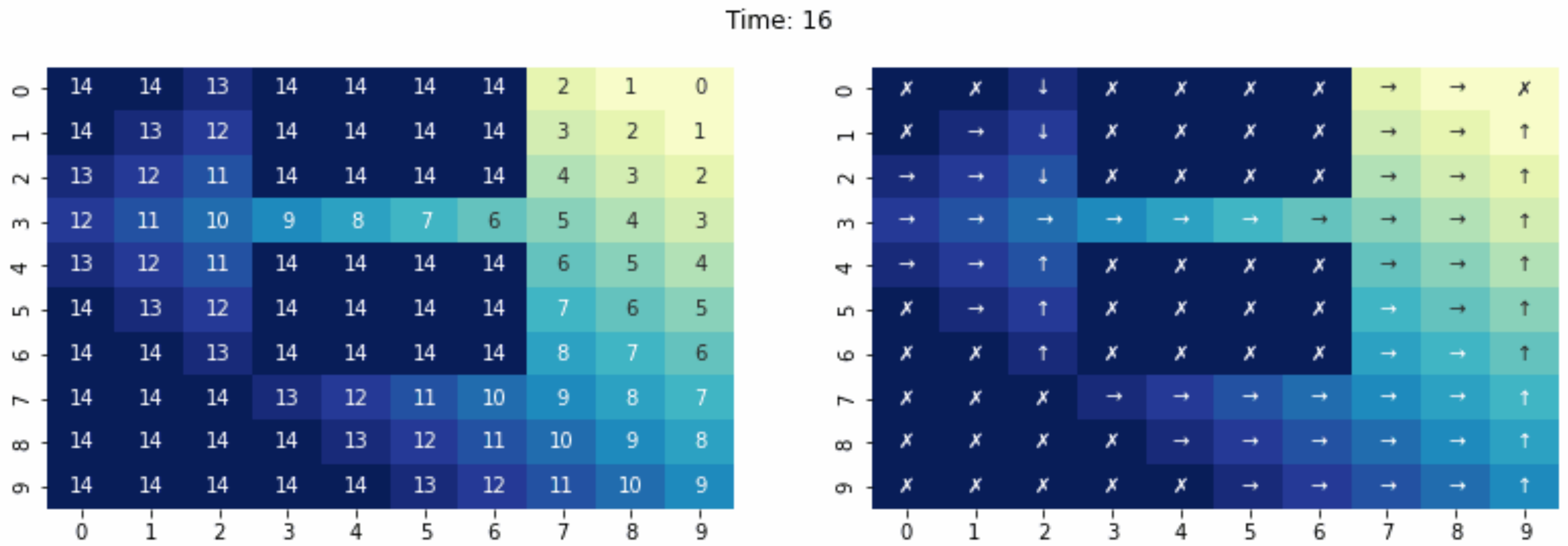
$$V^*(s_{T-1}) = \min_a c(s_{T-1}, a) \qquad \pi^*(s_{T-1}) = \arg\min_a c(s_{T-1}, a)$$

What is the optimal value at T-2?



$$V^*(s_{T-2}) = \min_a [c(s_{T-2}, a) + V^*(s_{T-1})] \qquad \pi^*(s_{T-2}) = \arg\min_a [c(s_{T-2}, a) + V^*(s_{T-1})]$$

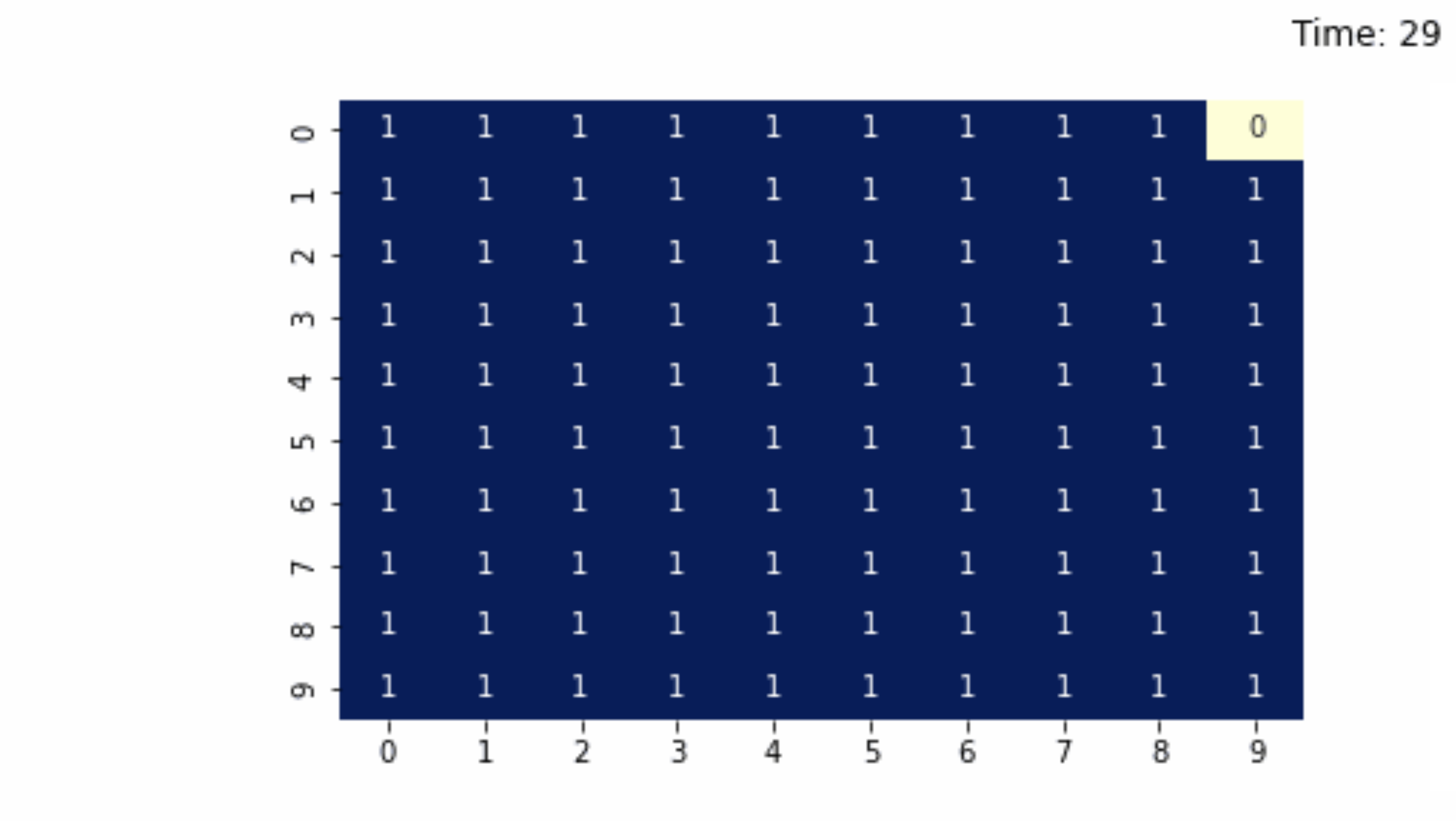
Dynamic Programming all the way!



$$V^*(s_t) = \min_a [c(s_t, a) + V^*(s_{t+1})]$$

$$\pi^*(s_t) = \arg\min_a [c(s_t), a) + V^*(s_{t+1})]$$

Value Iteration



Algorithm 4: Dynamic Programming Value Iteration for computing the optimal value function.

Algorithm `OptimalValue`(x, T)
- **for** $t = T-1, \ldots, 0$ **do**
  - **for** $x \in \mathbb{X}$ **do**
    - **if** $t = T-1$ **then**
      - $V(x,t) = \min_a c(x,a)$
    - **end**
    - **else**
      - $V(x,t) = \min_a c(x,a) + \sum_{x' \in \mathbb{X}} p(x'|x,a) V(x, t+1)$
    - **end**
  - **end**
- **end**

What is the complexity?

$S \times A \times T$ — Deterministic

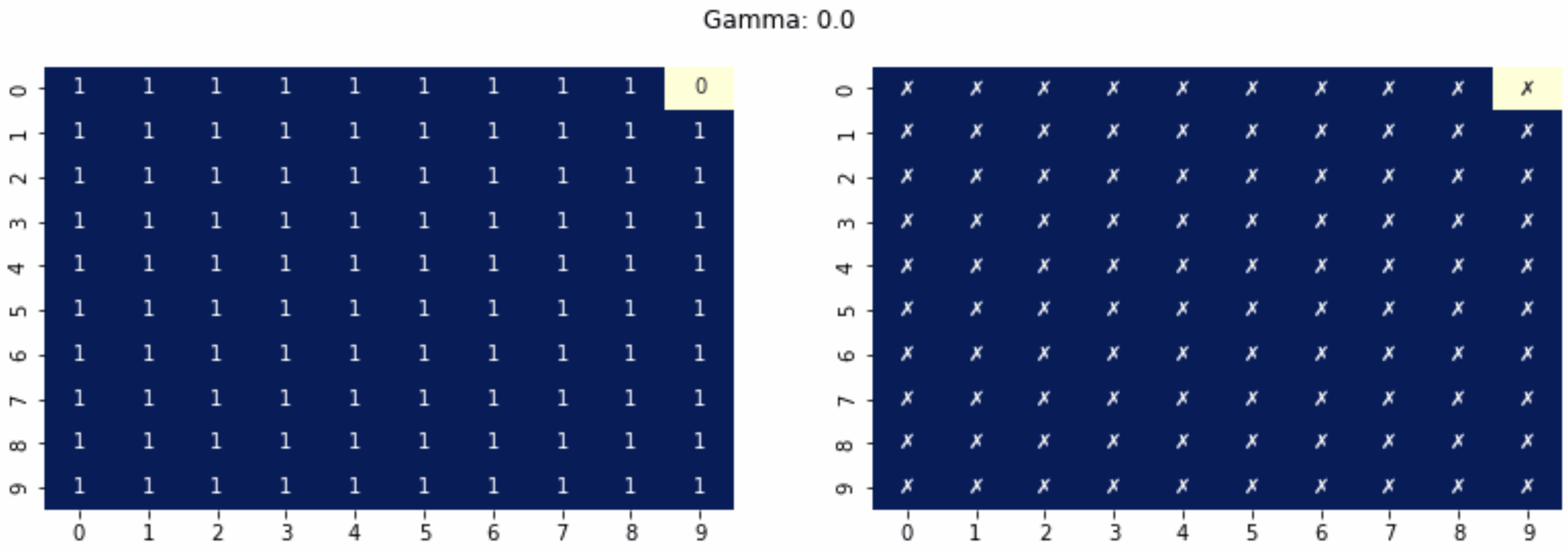
$S^2 \times A \times T$ — Stochastic

$k \times S \times A \times T$ — Efficient

Why is the optimal policy a function of time?

Pulling the goalie
when you
are losing and have
seconds left ..

What is the effect of discount factor?



Many questions!

Q1. What about continuous MDPs?

Next class :)

Q2. What if my horizon was infinite?

$$V^*(s_t) = \min_{a_t} \left[c(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1} \sim \mathcal{T}(s_t, a_t)} V^*(s_{t+1}) \right] \xrightarrow{\text{(Fixed point)}} V^*(s) = \min_{a} \left[c(s, a) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s, a)} V^*(s) \right]$$

Q3. Is value iteration the only way?

No, but it will give us some mileage for now.
Will cover policy iteration later!

To infinity!

Infinite horizon cases

$$V^*(s_t) = \min_{a_t} [c(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1} \sim \mathcal{T}(s_t, a_t)} V^*(s_{t+1})]$$

↓ Fixed point as $t \to \infty$

$$V^*(s) = \min_{a} [c(s, a) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s, a)} V^*(s)]$$

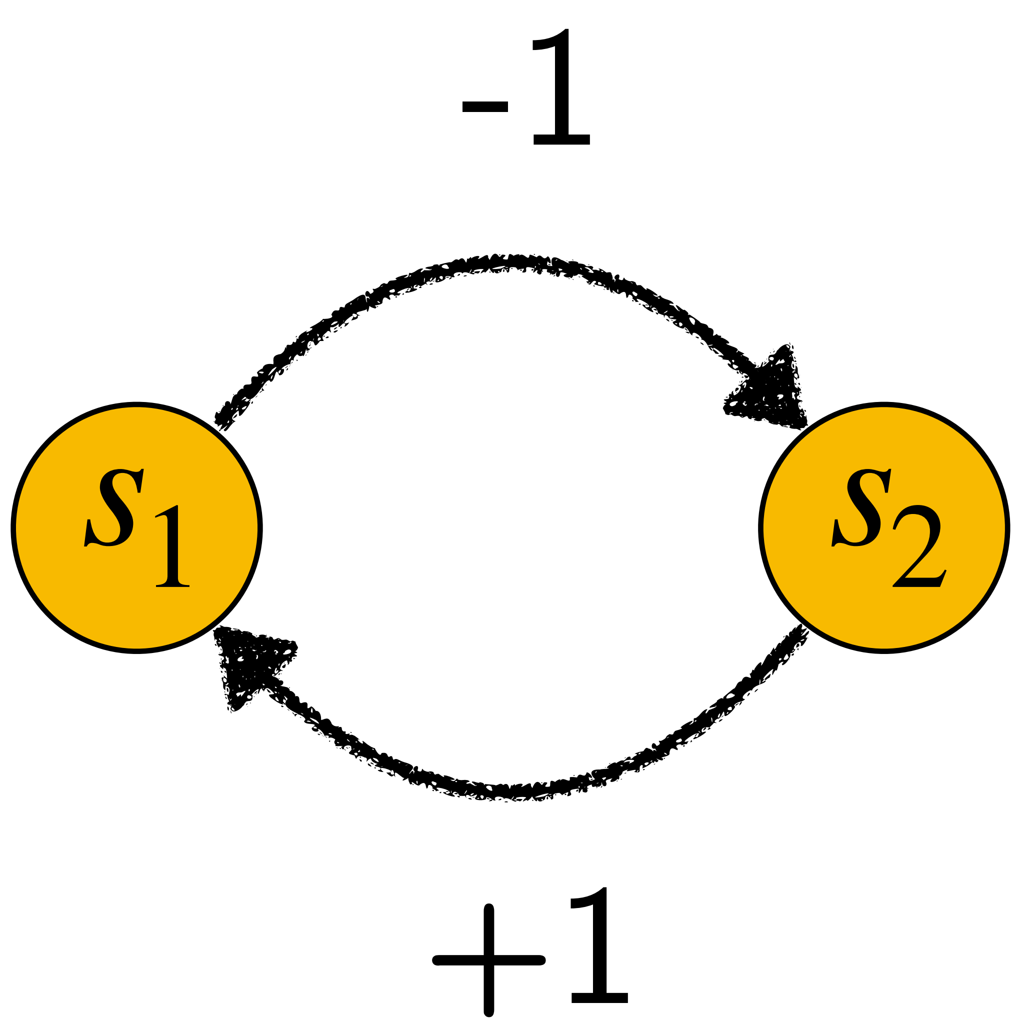
Bellman Equation

$$V^*(s) = \min_a \left[c(s, a) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^*(s)\right]$$

Does this converge?

How fast does it converge?

Does value iteration converge?



$\gamma = 1$

What is $V^*(s_1)$? What is $V^*(s_2)$?

tl;dr

Markov Decision Process

A mathematical framework for modeling sequential decision making

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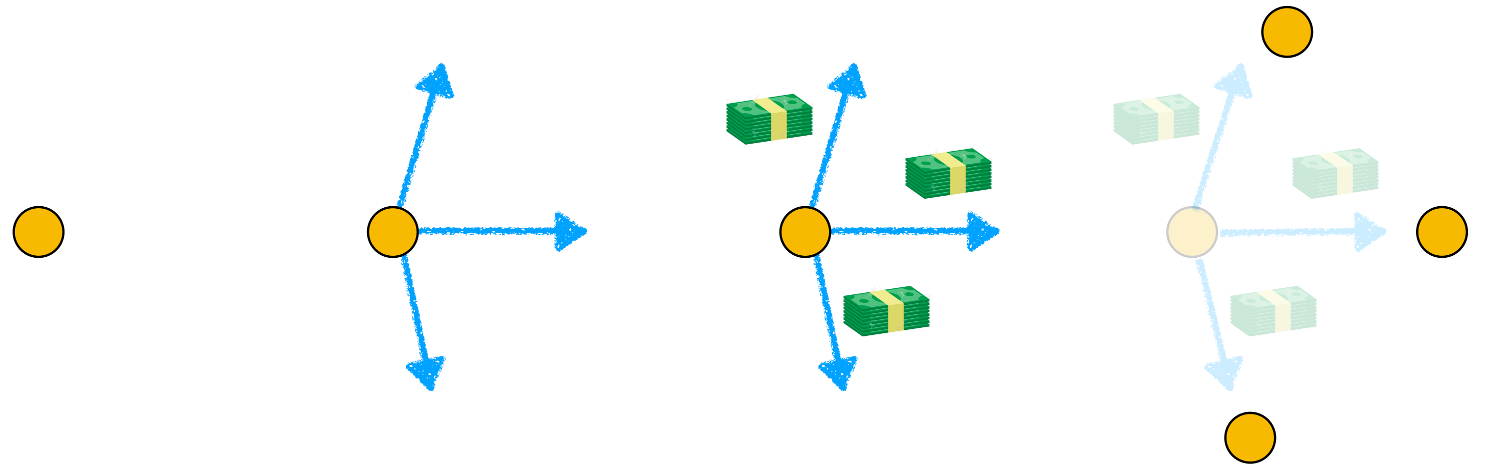
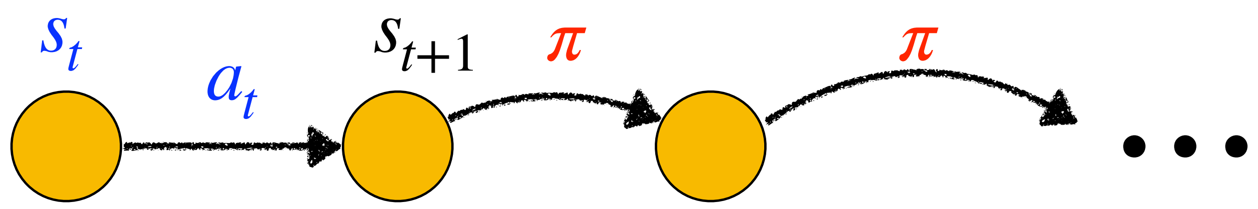


x

Value of a state-action



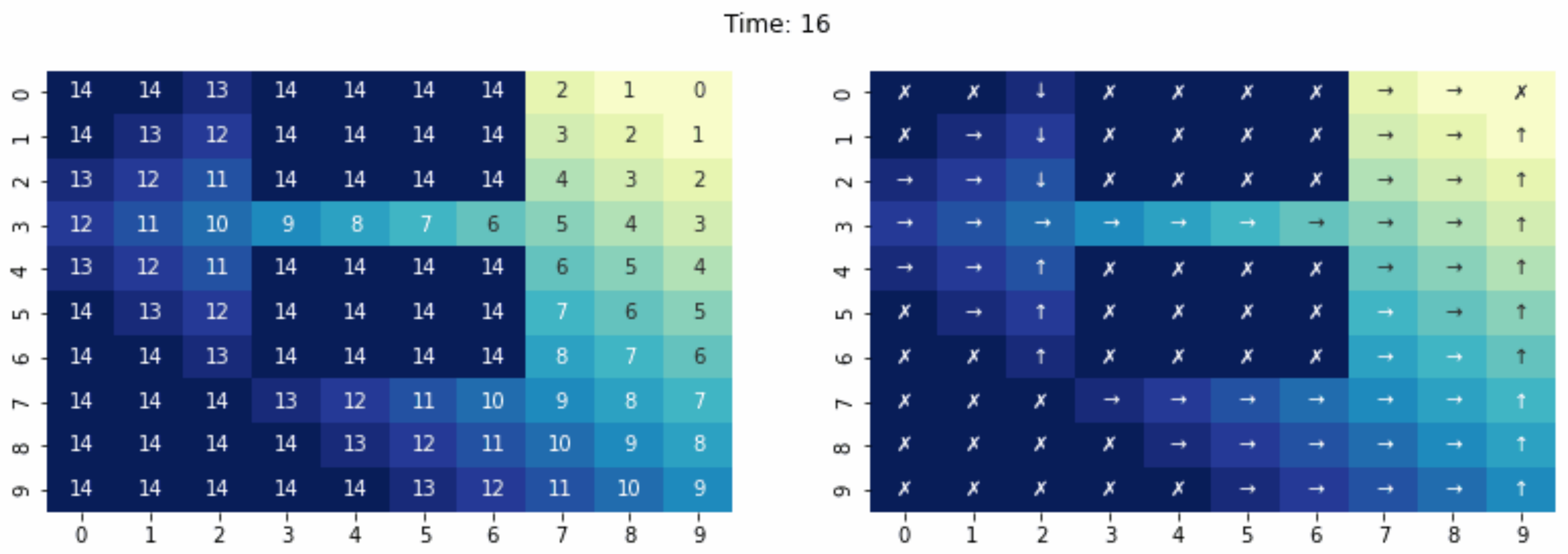
$$Q^{\pi}(s_t, a_t) = c_t + \gamma c_{t+1} + \gamma^2 c_{t+2} + \cdots$$

Expected discounted sum of cost from starting at a state, executing action and following a policy from then on

$$Q^{\pi}(s_t, a_t) = c(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1} \sim \mathcal{T}(s_t, a_t)} V^{\pi}(s_{t+1})$$

x

Dynamic Programming all the way!



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