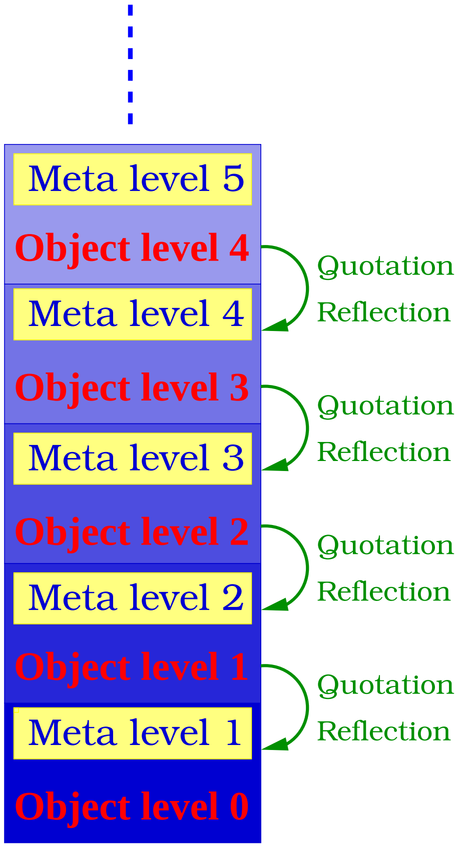
CS 671 Automated Reasoning

Reflection

Reflection – basic methodology

- **Represent object and meta level in type theory**
  - Represent meta-logical concepts as Nuprl terms
  - Express specific object logic in represented meta logic
  - Build hierarchy: level i contains meta level for level $i{+}1$
  - $\mapsto$ Reasoning about both levels from the "outside"



- **Link object logic and meta-logic**
  - Embed object level terms using quotation
  - Embed object level provability using reflection rule

- **Use same reasoning apparatus for object and meta level**

Reflection, technically (1)

- **Represent object level terms**

```
Term    ≡  rectype Term = Atom × Parm list × (Var list × Term) list
x       ≡  <<"variable", [x:v]> []>
λx.t    ≡  <<"lambda", []> [[x], t]>
(f t)   ≡  <<"apply", []> [[], f;[], t]>
```

- **Represent meta level operators**

```
subst :     Term -> Var -> Term -> Term
evalto:     Term -> Term
canonical:  Term -> 𝔹
in:         Term -> Term -> ℙ
                ⋮
```

- **Represent the proof theory**

```
Sequent  ≡  (Var × Term)list × Term
Proof    ≡  Dequent × Rule × Proof list
```

Reflection, technically (2)

- **Prove semantical relationships**

$$\begin{array}{lll}
\texttt{Term} & \doteq & \texttt{term} \\
\lceil t \rceil \;\texttt{in}\; \lceil T \rceil & \doteq & t \in T \\
\texttt{Proof} & \doteq & \texttt{proof} \\
\lceil t_1 \rceil \;\texttt{evalto}\; \lceil t_2 \rceil & \doteq & t_1 \downarrow t_2 \\
\exists \texttt{p:Proof.goal(p) = } \lceil H \vdash A \rceil & \doteq & H \vdash A \quad \text{is valid}
\end{array}$$

- **Add reflection rule**

$$H \vdash_{i+1} A \qquad \textbf{by } \texttt{reflection } i$$
$$\vdash_i \exists \texttt{p:Proof}_i\texttt{. goal(p) = } \lceil H \vdash_{i+1} A \rceil$$

- **Prove that reflection does not change logic**
  – If a sequent s is provable then it is provable without reflection

See "*The Semantics of Reflected Proof*", (S. Allen, R. Constable, D. Howe, W. Aitken, 1990)

WHY LEVELS OF REFLECTION?

Can we use naive reflection ?

$H \vdash A$ **by** `reflection`

$H \vdash \exists \mathtt{p{:}Proof.\ goal(p)} = \lceil \vdash A \rceil$

This would enable us to prove

$\vdash \neg(\exists \mathtt{p{:}Proof.goal(p)} = \lceil \vdash \mathtt{False} \rceil)$

`BY notR`

$\exists \mathtt{p{:}Proof.goal(p)} = \lceil \vdash \mathtt{False} \rceil \vdash \mathtt{False}$

`BY reflection`

$\exists \mathtt{p{:}Proof.goal(p)} = \lceil \vdash \mathtt{False} \rceil \vdash \exists \mathtt{p{:}Proof.goal(p)} = \lceil \vdash \mathtt{False} \rceil$

`BY hypotheses`

But Gödel's second incompleteness theorem states

If a consistent, axiomatizable theory $\mathcal{T}$ subsumes arithmetic, then it is impossible to prove the consistency of $\mathcal{T}$ within $\mathcal{T}$

Why levels of reflection?

What if we require all hypotheses to be reflected?

$H \vdash A$ **by** `reflection`

$\vdash \exists$`p:Proof. goal(p) =` $\lceil H \vdash A \rceil$

If this rule does not change the logic we should be able to prove

$\vdash$ ($\exists$`p:Proof. goal(p) =` $\lceil H \vdash A \rceil$) $\Rightarrow$ ($H \Rightarrow A$)

without the reflection rule, which violates Gödel's theorem.

Adding a reflection rule leads to a hierarchy of proof levels, which may not be closed off proof theoretically.
The reflection rule must include indices to separate the levels.

See "*Metaprogramming in Nuprl using Reflection*" (W. Aitken, PHD Thesis 1994)

Reflection in practice

- **Reflection leads to blow-up of term size**
  – Small terms represented by large tuples
- **Abstractions and display forms can reduce blow-up**
  – Prove laws of reflected concepts and terms
  – Don't unfold definitions in formal reasoning
  – Use colors in displays to separate levels
- **Substitution and computation remain inefficient**
  – Mechanisms have to be simulated to avoid unfolding terms
  – Can't use built-in mechanisms

Reflecting terms more efficiently (E. Barzilay)

- **Change the internal representation of Nuprl terms**
  - Include quotation level as additional parameter of every term
  - All object levels use the same term syntax

$$
\begin{array}{lcl}
x & \doteq & \texttt{variable } \{x\texttt{:v, 0:Q}\}() \\
\lambda x.t & \doteq & \texttt{lambda } \{\texttt{0:Q}\}(x.t) \\
(f\ t) & \doteq & \texttt{apply } \{\texttt{0:Q}\}(f;t) \\
x & \doteq & \texttt{variable } \{x\texttt{:v, 1:Q}\}() \\
\lambda x.t & \doteq & \texttt{lambda } \{\texttt{1:Q}\}(x.t) \\
(f\ t) & \doteq & \texttt{apply } \{\texttt{1:Q}\}(f;t)
\end{array}
$$

  - Some technical subtleties: mixed quotation levels, quoted bindings, ...

- **Use built-in substitution and computation**

- **Extend type theory by quotation operator $[[t]]$**
  - Meaning $[[t]]$ of t is the obvious term of the next quotation level below

$$[[opid\{p_i{:}F_i, j\texttt{+1:Q}\}(subterms)]] = opid\{p_i{:}F_i, j\texttt{:Q}\}(subterms)$$

  - Define operators `subst`, `evalto`, `canonical`, `in`, ... using $[[t]]$

- **Reflection of other concepts almost straightforward**

Applications

- **Improving proof automation in theorem proving**
  - Enable proofs by syntactical checks
- **Formal proof theory**
  - Elegant accounts of Gödel's theorems, ...
- **Reasoning about program transformations**
  - Optimizations, aspect weaving
- **Reasoning about computational complexity**
  - Complexity classes
  - Resource-bounded logic