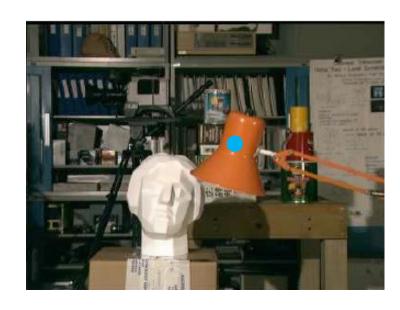
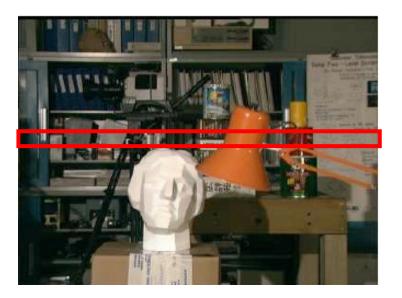
Reconstruction

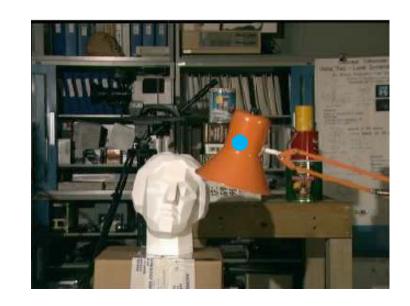
Perspective projection in rectified cameras

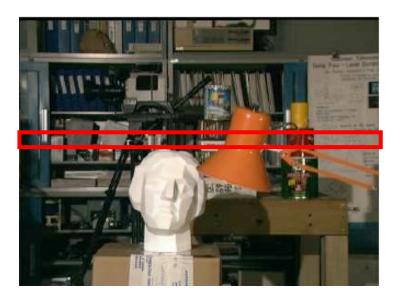




- For rectified cameras, correspondence problem is easier
- Only requires searching along a particular row.

Epipolar constraint





 Reduces 2D search problem to search along a particular line: epipolar line

Epipolar constraint

True in general!

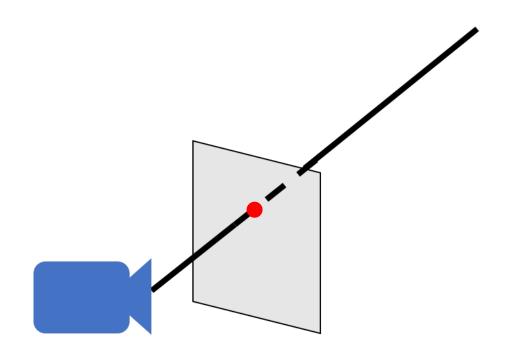
- Given pixel (x,y) in one image, corresponding pixel in the other image must lie on a line
- Line function of (x,y) and parameters of camera
- These lines are called *epipolar line*



Epipolar geometry

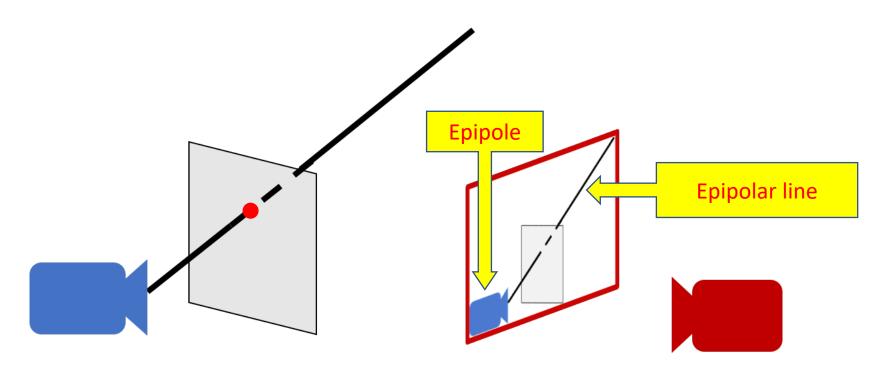
Epipolar geometry - why?

• For a single camera, pixel in image plane must correspond to point somewhere along a ray



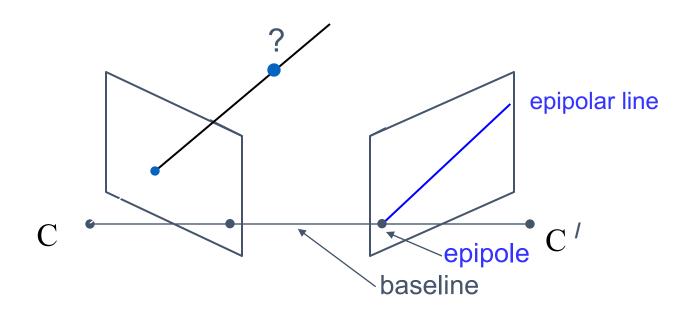
Epipolar geometry

- When viewed in second image, this ray looks like a line: epipolar line
- The epipolar line must pass through image of the first camera in the second image epipole



Epipolar geometry

Given an image point in one view, where is the corresponding point in the other view?



- A point in one view "generates" an epipolar line in the other view
- The corresponding point lies on this line

Epipolar line



Epipolar constraint

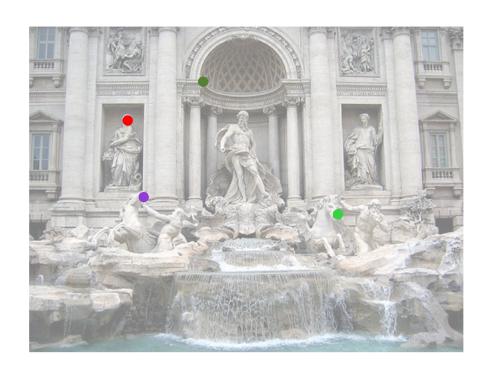
• Reduces correspondence problem to 1D search along an epipolar line

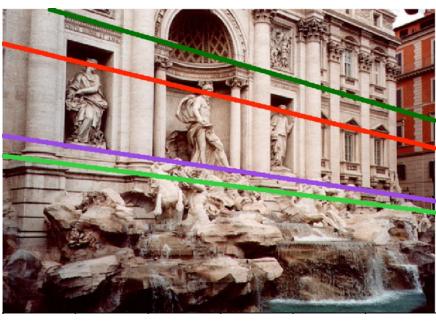
Epipolar lines



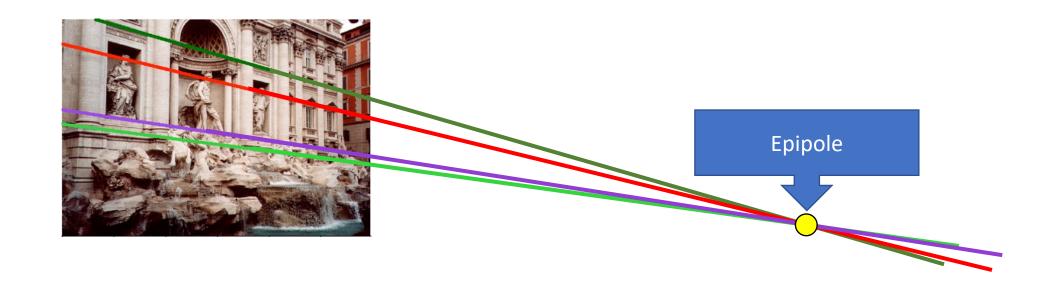


Epipolar lines



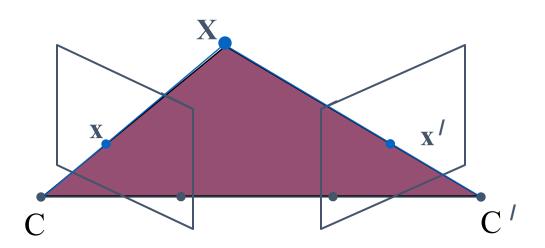


Epipolar lines



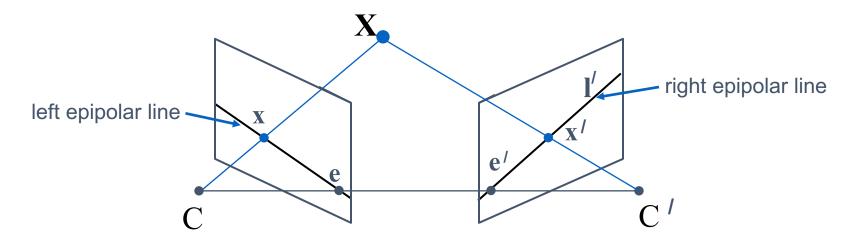
Epipolar geometry continued

Epipolar geometry is a consequence of the coplanarity of the camera centres and scene point



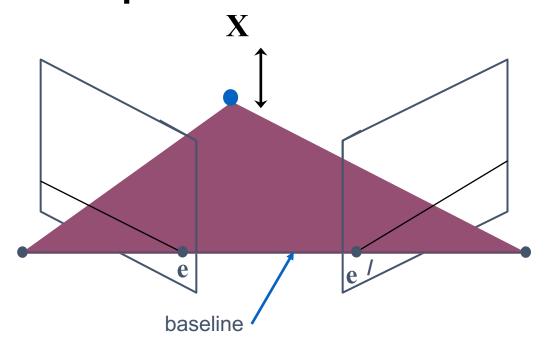
The camera centres, corresponding points and scene point lie in a single plane, known as the epipolar plane

Nomenclature



- The epipolar line \mathbf{l}' is the image of the ray through \mathbf{x}
- The epipole e is the point of intersection of the line joining the camera centres with the image plane
 - this line is the baseline for a stereo rig, and
 - the translation vector for a moving camera
- The epipole is the image of the centre of the other camera: $\mathbf{e} = \mathbf{PC}'$, $\mathbf{e}' = \mathbf{P'C}$

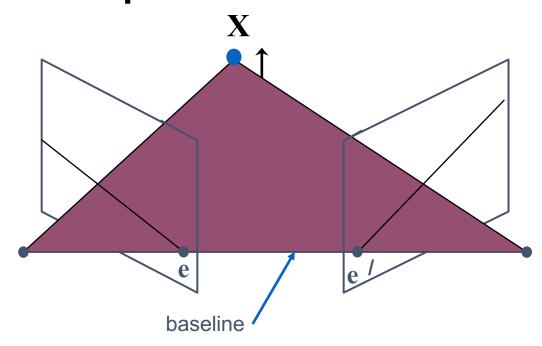
The epipolar pencil



As the position of the 3D point X varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil (a pencil is a one parameter family).

All epipolar lines intersect at the epipole.

The epipolar pencil



As the position of the 3D point X varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil (a pencil is a one parameter family).

All epipolar lines intersect at the epipole.

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv K_1 \begin{bmatrix} R_1 & \mathbf{t}_1 \end{bmatrix} \vec{\mathbf{x}}_w$$
$$\vec{\mathbf{x}}_{img}^{(2)} \equiv K_2 \begin{bmatrix} R_2 & \mathbf{t}_2 \end{bmatrix} \vec{\mathbf{x}}_w$$

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv \begin{bmatrix} I & 0 \end{bmatrix} \vec{\mathbf{x}}_w$$
$$\vec{\mathbf{x}}_{img}^{(2)} \equiv \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} = \mathbf{x}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} = R\mathbf{x}_w + \mathbf{t}$$

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv \mathbf{x}_w$$
 $\vec{\mathbf{x}}_{img}^{(2)} \equiv R\mathbf{x}_w + \mathbf{t}$

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\lambda_1 \vec{\mathbf{x}}_{img}^{(1)} = \mathbf{x}_w$$

$$\lambda_2 \vec{\mathbf{x}}_{imq}^{(2)} = R\mathbf{x}_w + \mathbf{t}$$

$$\lambda_{2}\vec{\mathbf{x}}_{img}^{(2)} = \lambda_{1}R\vec{\mathbf{x}}_{img}^{(1)} + \mathbf{t}$$

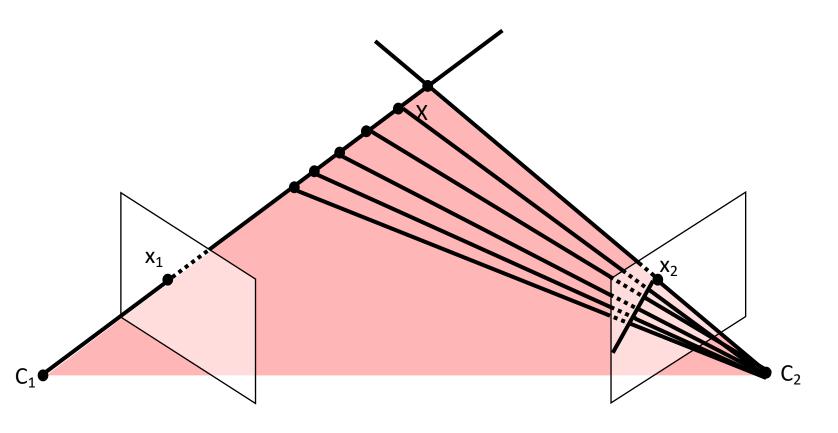
$$\lambda_{2}\mathbf{t} \times \vec{\mathbf{x}}_{img}^{(2)} = \lambda_{1}\mathbf{t} \times R\vec{\mathbf{x}}_{img}^{(1)} + \mathbf{t} \times \mathbf{t}$$

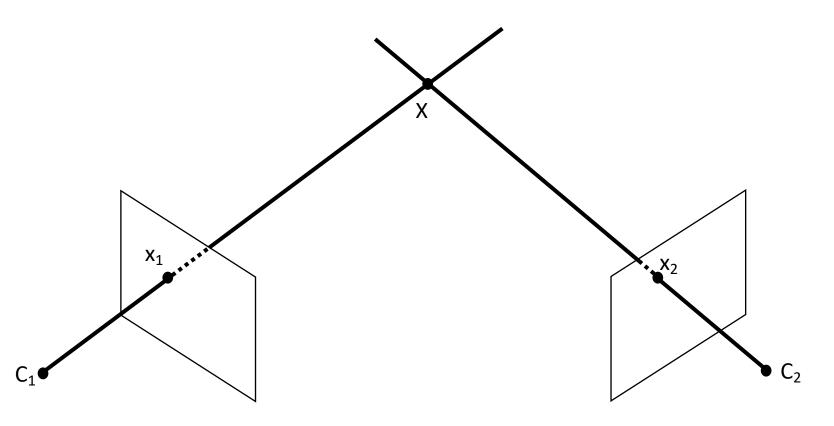
$$\lambda_{2}\mathbf{t} \times \vec{\mathbf{x}}_{img}^{(2)} = \lambda_{1}\mathbf{t} \times R\vec{\mathbf{x}}_{img}^{(1)}$$

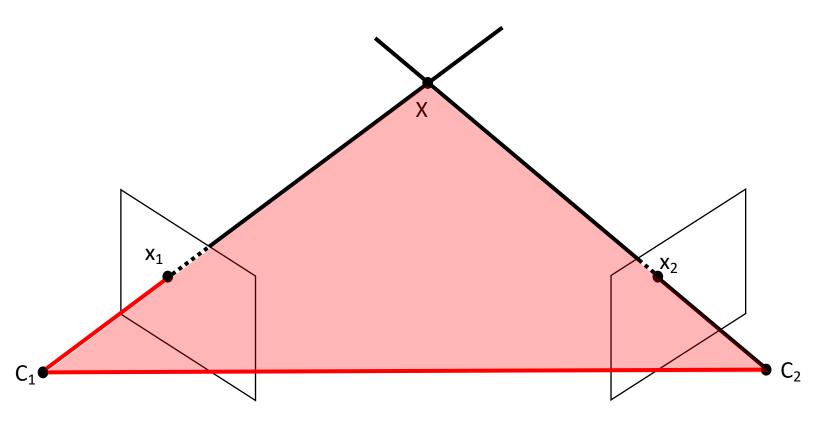
$$\lambda_{2}\mathbf{t} \times \vec{\mathbf{x}}_{img}^{(2)} = \lambda_{1}\mathbf{t} \times R\vec{\mathbf{x}}_{img}^{(1)}$$

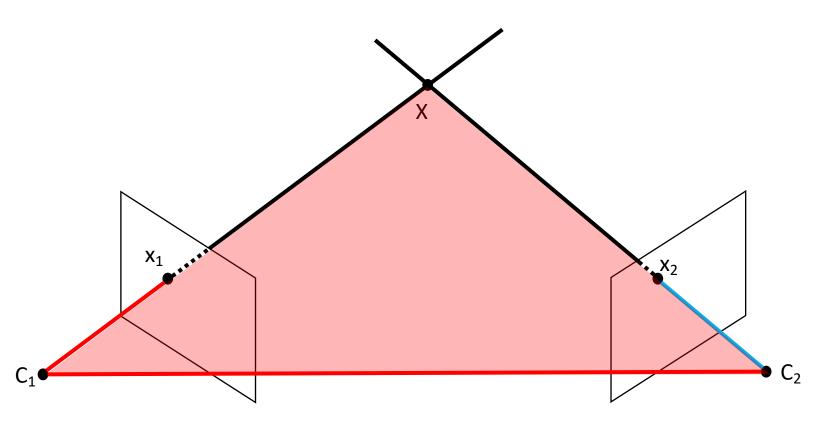
$$\lambda_{2}\vec{\mathbf{x}}_{img}^{(2)} \cdot \mathbf{t} \times \vec{\mathbf{x}}_{img}^{(2)} = \lambda_{1}\vec{\mathbf{x}}_{img}^{(2)} \cdot \mathbf{t} \times R\vec{\mathbf{x}}_{img}^{(1)}$$

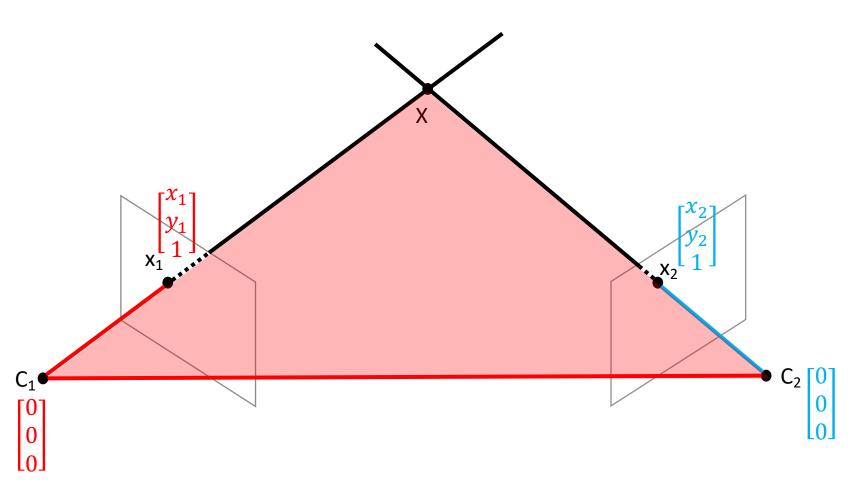
$$0 = \lambda_{1}\vec{\mathbf{x}}_{img}^{(2)} \cdot \mathbf{t} \times R\vec{\mathbf{x}}_{img}^{(1)}$$

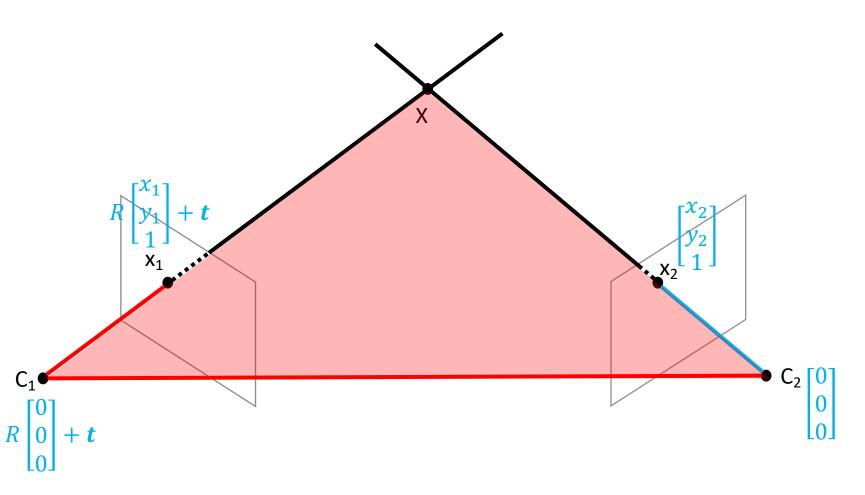


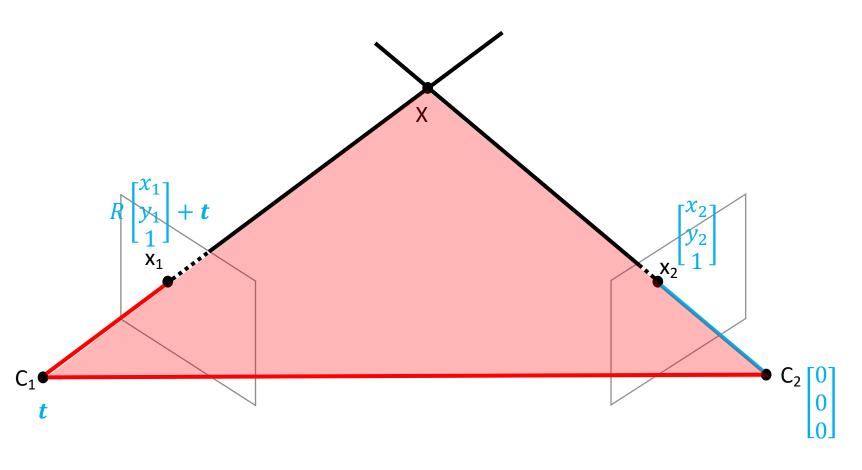


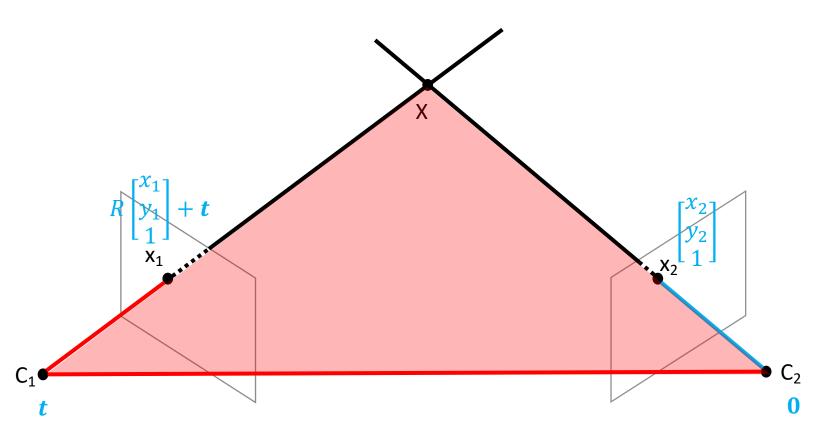


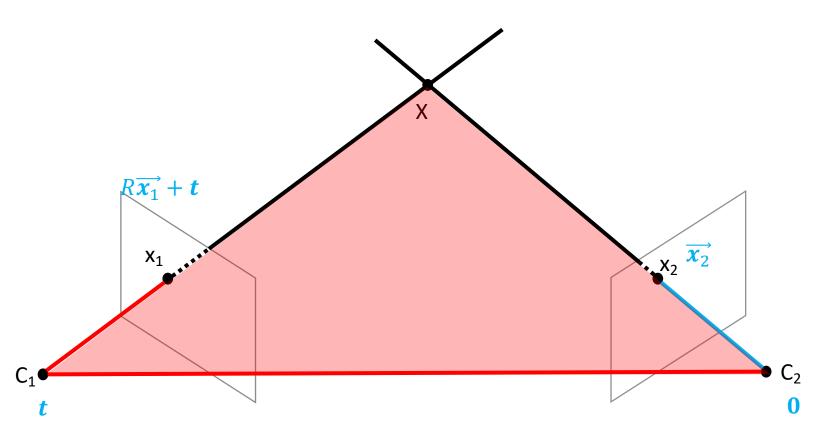


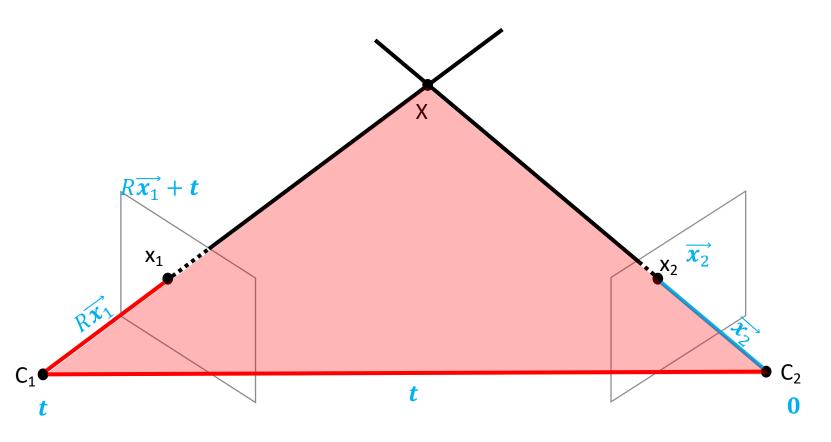


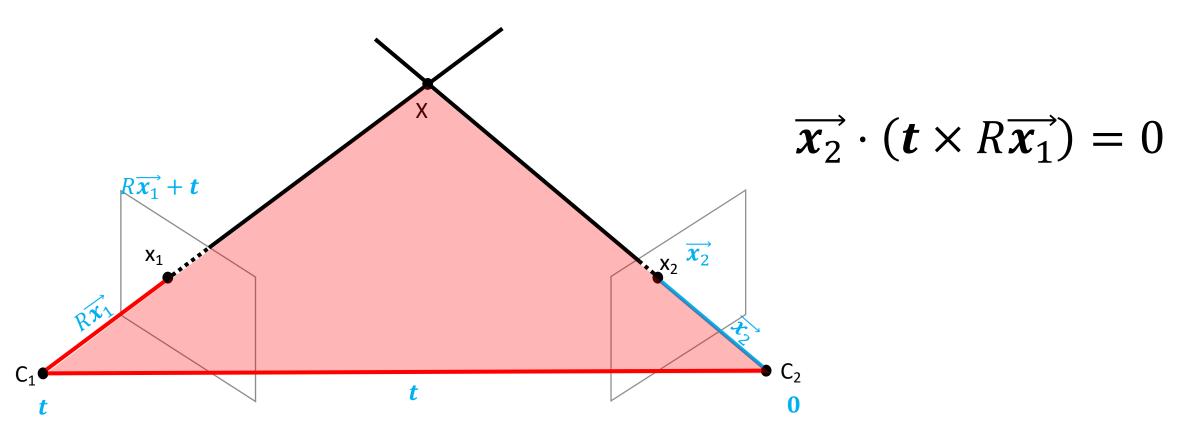












$$\vec{\mathbf{x}}_{img}^{(2)} \cdot \mathbf{t} \times R \vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Can we write this as matrix vector operations?
- Cross product can be written as a matrix

$$egin{aligned} [\mathbf{t}]_{ imes} &= egin{bmatrix} 0 & -t_z & t_y \ t_z & 0 & -t_x \ -t_y & t_x & 0 \end{bmatrix} \ [\mathbf{t}]_{ imes} \mathbf{a} &= \mathbf{t} imes \mathbf{a} \end{aligned}$$

$$\vec{\mathbf{x}}_{img}^{(2)} \cdot [\mathbf{t}]_{\times} R \vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Can we write this as matrix vector operations?
- Dot product can be written as a vector-vector times

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$$

$$\vec{\mathbf{x}}_{img}^{(2)} \cdot [\mathbf{t}]_{\times} R \vec{\mathbf{x}}_{img}^{(1)} = 0$$

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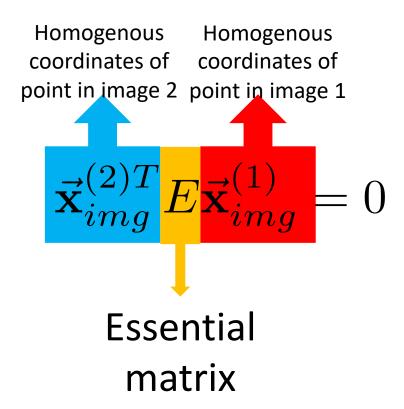
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$$

Epipolar geometry - the math

$$\vec{\mathbf{x}}_{img}^{(2)T}[\mathbf{t}]_{\times}R\vec{\mathbf{x}}_{img}^{(1)} = 0$$

$$\vec{\mathbf{x}}_{img}^{(2)T}E\vec{\mathbf{x}}_{img}^{(1)} = 0$$

Epipolar geometry - the math



Epipolar constraint and epipolar lines

$$\vec{\mathbf{x}}_{img}^{(2)T} E \vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Consider a known, fixed pixel in the first image
- What constraint does this place on the corresponding pixel?

•
$$\vec{\mathbf{x}}_{img}^{(2)T}\mathbf{l} = 0$$
 where $\mathbf{l} = E\vec{\mathbf{x}}_{img}^{(1)}$

What kind of equation is this?

Epipolar constraint and epipolar lines

$$\vec{\mathbf{x}}_{img}^{(2)T} E \vec{\mathbf{x}}_{img}^{(1)} = 0$$

Consider a known, fixed pixel in the first image

•
$$\vec{\mathbf{x}}_{img}^{(2)T}\mathbf{l} = 0$$
 where $\mathbf{l} = E\vec{\mathbf{x}}_{img}^{(1)}$

$$\vec{\mathbf{x}}_{img}^{(2)T}\mathbf{l} = 0$$

$$\Rightarrow \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} = 0$$

$$\Rightarrow l_x x_2 + l_y y_2 + l_z = 0$$
Line!

Epipolar constraint: putting it all together

• If **p** is a pixel in first image and **q** is the corresponding pixel in the second image, then:

$$\mathbf{q}^{\mathsf{T}}\mathbf{E}\mathbf{p}=0$$

- $E = [t]_X R$
- For fixed \mathbf{p} , \mathbf{q} must satisfy: $\mathbf{q}^T \mathbf{l} = 0$, where $\mathbf{l} = \mathbf{E}\mathbf{p}$
- For fixed \mathbf{q} , \mathbf{p} must satisfy: $\mathbf{I}^T \mathbf{p} = 0$ where $\mathbf{I}^T = \mathbf{q}^T \mathbf{E}$, or $\mathbf{I} = \mathbf{E}^t \mathbf{q}$
- These are epipolar lines!

Epipolar line in 2nd image

Epipolar line in 1st image

Essential matrix and epipoles

• $E = [t]_X R$

$$\vec{\mathbf{c}}_2 = \mathbf{t}$$
 $\vec{\mathbf{c}}_2^T E = \mathbf{t}^T E = \mathbf{t}^T [\mathbf{t}]_{\times} R = 0$
 $\vec{\mathbf{c}}_2^T E \mathbf{p} = 0 \ \forall \mathbf{p}$

- Ep is an epipolar line in 2nd image
- All epipolar lines in second image pass through c₂
- c₂ is epipole in 2nd image

Essential matrix and epipoles

•
$$\mathbf{E} = [\mathbf{t}]_{\mathsf{X}} \mathbf{R}$$

$$\vec{\mathbf{c}}_{1} = \mathbf{R}^{T} \mathbf{t}$$

$$E\vec{\mathbf{c}}_{1} = [\mathbf{t}]_{\mathsf{X}} R R^{T} \mathbf{t} = [\mathbf{t}]_{\mathsf{X}} \mathbf{t} = 0$$

$$\mathbf{q}^{T} E \vec{\mathbf{c}}_{1} = 0 \quad \forall \mathbf{q}$$

- E^T**q** is an epipolar line in 1st image
- All epipolar lines in first image pass through c₁
- c₁ is the epipole in 1st image

Epipolar geometry - the math

- We assumed that intrinsic parameters K are identity
- What if they are not?

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv K_1 \begin{bmatrix} R_1 & \mathbf{t}_1 \end{bmatrix} \vec{\mathbf{x}}_w$$
$$\vec{\mathbf{x}}_{img}^{(2)} \equiv K_2 \begin{bmatrix} R_2 & \mathbf{t}_2 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv K_1 \begin{bmatrix} I & 0 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv K_2 \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\lambda_1 \vec{\mathbf{x}}_{img}^{(1)} = K_1 \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \vec{\mathbf{x}}_w$$
$$\lambda_2 \vec{\mathbf{x}}_{img}^{(2)} = K_2 \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\lambda_{1}\vec{\mathbf{x}}_{img}^{(1)} = K_{1} \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \vec{\mathbf{x}}_{w}$$

$$= K_{1} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{w} \\ 1 \end{bmatrix}$$

$$= K_{1}\mathbf{x}_{w}$$

$$\Rightarrow \lambda_1 K_1^{-1} \vec{\mathbf{x}}_{img}^{(1)} = \mathbf{x}_w$$

$$\lambda_{2}\vec{\mathbf{x}}_{img}^{(2)} = K_{2} \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{w} \\ 1 \end{bmatrix}$$

$$= K_{2}R\mathbf{x}_{w} + K_{2}\mathbf{t}$$

$$= \lambda_{1}K_{2}RK_{1}^{-1}\vec{\mathbf{x}}_{img}^{(1)} + K_{2}\mathbf{t}$$

$$\Rightarrow \lambda_{2}K_{2}^{-1}\vec{\mathbf{x}}_{img}^{(2)} = \lambda_{1}RK_{1}^{-1}\vec{\mathbf{x}}_{img}^{(1)} + \mathbf{t}$$

$$\Rightarrow \lambda_{2}[\mathbf{t}]_{\times}K_{2}^{-1}\vec{\mathbf{x}}_{img}^{(2)} = \lambda_{1}[\mathbf{t}]_{\times}RK_{1}^{-1}\vec{\mathbf{x}}_{img}^{(1)}$$

$$\Rightarrow 0 = \vec{\mathbf{x}}_{img}^{(2)}K_{2}^{-T}[\mathbf{t}]_{\times}RK_{1}^{-1}\vec{\mathbf{x}}_{img}^{(1)}$$

$$\Rightarrow 0 = \vec{\mathbf{x}}_{img}^{(2)} K_2^{-T} [\mathbf{t}]_{\times} R K_1^{-1} \vec{\mathbf{x}}_{img}^{(1)}$$

$$\Rightarrow 0 = \vec{\mathbf{x}}_{img}^{(2)} F \vec{\mathbf{x}}_{img}^{(1)}$$

Fundamental matrix result

$$\mathbf{q}^T \mathbf{F} \mathbf{p} = 0$$

(Longuet-Higgins, 1981)

Properties of the Fundamental Matrix

• ${f F}_{f D}$ s the epipolar line associated with ${f p}$

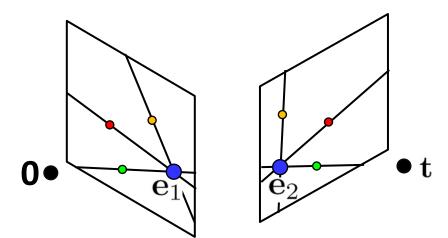
 $oldsymbol{F}^T \mathbf{q}$ s the epipolar line associated with

 \mathbf{e}_1

 \mathbf{q}

Properties of the Fundamental Matrix

- f Fpis the epipolar line associated with f P
- ullet $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with \mathbf{q}
- $oldsymbol{\cdot} \mathbf{F} \mathbf{e}_1 = \mathbf{0} \ ext{and} \ \mathbf{F}^T \mathbf{e}_2 = \mathbf{0}$
- All epipolar lines contain epipole

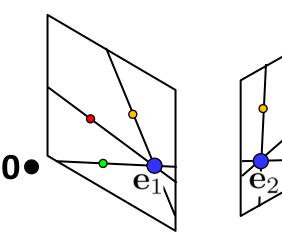


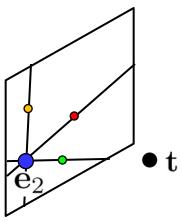
Properties of the Fundamental Matrix

- $oldsymbol{\cdot}$ $oldsymbol{\mathbf{F}}_{oldsymbol{p}}$ is the epipolar line associated with $oldsymbol{p}$
- $oldsymbol{\cdot}$ $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with \mathbf{q}

$$oldsymbol{\cdot}$$
 $\mathbf{F}\mathbf{e}_1=\mathbf{0}$ and $\mathbf{F}^T\mathbf{e}_2=\mathbf{0}$

• \mathbf{F} is rank 2





Why is F rank 2?

- F is a 3 x 3 matrix
- But there is a vector c_1 and c_2 such that $Fc_1 = 0$ and $F^Tc_2 = 0$

Estimating F





- If we don't know **K**₁, **K**₂, **R**, or **t**, can we estimate **F** for two images?
- Yes, given enough correspondences

Estimating F – 8-point algorithm

The fundamental matrix F is defined by

$$\mathbf{x'}^{\mathsf{T}}\mathbf{F}\mathbf{x} = \mathbf{0}$$

for any pair of matches x and x' in two images.

• Let
$$\mathbf{x} = (u, v, 1)^{\mathsf{T}}$$
 and $\mathbf{x}' = (u', v', 1)^{\mathsf{T}}$,
$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

each match gives a linear equation

$$uu'f_{11} + vu'f_{12} + u'f_{13} + uv'f_{21} + vv'f_{22} + v'f_{23} + uf_{31} + vf_{32} + f_{33} = 0$$

8-point algorithm

$$\begin{bmatrix} u_{1}u_{1}' & v_{1}u_{1}' & u_{1}v_{1}' & v_{1}v_{1}' & v_{1}' & u_{1} & v_{1} & 1 \\ u_{2}u_{2}' & v_{2}u_{2}' & u_{2}' & u_{2}v_{2}' & v_{2}v_{2}' & v_{2}' & u_{2} & v_{2} & 1 \\ \vdots & \vdots \\ u_{n}u_{n}' & v_{n}u_{n}' & u_{n}' & u_{n}v_{n}' & v_{n}v_{n}' & v_{n}' & u_{n} & v_{n} & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

• In reality, instead of solving $\mathbf{Af} = 0$, we seek \mathbf{f} to minimize $\|\mathbf{Af}\|$, least eigenvector of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$.

8-point algorithm — Problem?

- F should have rank 2
- To enforce that **F** is of rank 2, F is replaced by F' that minimizes $\|\mathbf{F} \mathbf{F}'\|$ subject to the rank constraint.

• This is achieved by SVD. Let $\mathbf{F} = \mathbf{U}\Sigma\mathbf{V}$, where

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$
, let $\Sigma' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

then $\mathbf{F'} = \mathbf{U} \mathbf{\Sigma'} \mathbf{V}^{\mathrm{T}}$ is the solution.

Recovering camera parameters from F / E

• Can we recover R and t between the cameras from F?

$$F = K_2^{-T}[\mathbf{t}] \times RK_1^{-1}$$

- No: K₁ and K₂ are in principle arbitrary matrices
- What if we knew K₁ and K₂ to be identity?

$$E = [\mathbf{t}]_{\times} R$$

Recovering camera parameters from E

$$E = [\mathbf{t}]_{\times} R$$

$$\mathbf{t}^T E = \mathbf{t}^T [\mathbf{t}]_{\times} R = 0$$

$$E^T \mathbf{t} = 0$$

- **t** is a solution to E^T **x** = 0
- Can't distinguish between t and ct for constant scalar c
- How do we recover R?

Recovering camera parameters from E

$$E = [\mathbf{t}]_{\times} R$$

- We know E and t
- Consider taking SVD of E and [t]_X

$$[\mathbf{t}]_{\times} = U\Sigma V^{T}$$

$$E = U'\Sigma'V'^{T}$$

$$U'\Sigma'V'^{T} = E = [\mathbf{t}]_{\times}R = U\Sigma V^{T}R$$

$$U'\Sigma'V'^{T} = U\Sigma V^{T}R$$

$$V'^{T} = V^{T}R$$

Recovering camera parameters from E

$$E = [\mathbf{t}]_{\times} R$$

$$\mathbf{t}^T E = \mathbf{t}^T [\mathbf{t}]_{\times} R = 0$$

$$E^T \mathbf{t} = 0$$

- **t** is a solution to E^T **x** = 0
- Can't distinguish between t and ct for constant scalar c

8-point algorithm

- Pros: it is linear, easy to implement and fast
- Cons: susceptible to noise
- Degenerate: if points are on same plane

- Normalized 8-point algorithm: Hartley
 - Position origin at centroid of image points
 - Rescale coordinates so that center to farthest point is sqrt (2)