Volumetric Path Tracing

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[These notes are meant to supplement the lecture and the text, and they focus on writing down details clearly rather than on high-level explanations. In particular, the formulation of radiative transfer and the volumetric Rendering Equation is written down well in many places (for instance, Chandrasekhar's book, Ishimaru's book, Preisendorfer's monograph, our textbook, or other advanced rendering books).]

Path tracing can be applied to volumes just as well as it can be applied to surfaces. The resulting algorithm is pretty slow, though, especially for inhomogeneous or highly scattering media. As usual, the key to developing correct algorithms is to carefully write down the integrals involved and construct Monte Carlo estimators from them.

The integral form of the equation we need to solve, in the notation I've been using in lecture, is:

$$L(x,\omega) = \int_x^y \tau(x,x')\Big[\underbrace{\sigma_s(x')\int_{4\pi} f_p(x',\omega,\omega')\,L(x',\omega')\,d\omega'}_{\text{inscattering}} + \underbrace{\epsilon(x',\omega)}_{\text{emission}}\Big]\,dx' + \underbrace{\tau(x,y)\,L_e(y,\omega)}_{\text{attenuation}}$$

The MC approach is to identify the integrand, choose a point from a suitable pdf, and compute the ratio:

$$\frac{f}{p} = \tau(x,x')\Big[\sigma_s(x')\,f_p(x',\omega,\omega')\,L(x',\omega') + \frac{1}{4\pi}\,\epsilon(x',\omega)\Big]\Big/ p(x',\omega')$$

Since we will usually choose x' first, then choose ω', we can write this as a nested estimator that estimates the ray integral using an estimate for the scattering integral:

$$\frac{f}{p} = \tau(x,x')\left[\frac{\sigma_s(x')\,f_p(x',\omega,\omega')\,L(x',\omega')}{p(\omega')} + \epsilon(x',\omega)\right]\Big/ p(x')$$

Let's apply this first in some special cases.

1. Emission only

Here $\tau(x, x') = 1$, and $\sigma_s = \sigma_a = 0$. The integral is just

$$L(x,\omega) = \int_x^y \epsilon(x',\omega)\, dx'$$

If we use uniform sampling with respect to distance along the ray, we have the estimator:

$$g = \epsilon(x',\omega)\, \|x-y\|$$

If the emission is homogeneous, then the answer is simple: $\varepsilon\|x - y\|$.

2. Homogeneous, absorption only.

Here $\varepsilon = \sigma_s = 0$, and $\tau(x, y) = \exp(-\sigma_t\|x - y\|)$.

Since there is no scattering or emission the integral along the line goes away and we are left with the simple answer:

$L(x, \omega) = \tau(x, y)\, L_e(y, \omega)$.

3. Emission + homogeneous absorption.

Here $\sigma_s = 0$ so the remaining terms are:

$$L(x,\omega) = \int_x^y \tau(x,x')\, \epsilon(x',\omega)\, dx' + \tau(x,y)\, L_e(y,\omega)$$

For the integral,

$$f = \tau(x,x')\ \epsilon(x',\omega)$$

and for uniform sampling

$$g = \|x-y\|\, e^{-\sigma_a \|x-x'\|}\, \epsilon(x',\omega)$$

Once there is attenuation, uniform sampling is no longer a good strategy. We need to importance sample the attenuation.

Attenuation in the homogeneous medium is $\tau(d) = \exp(-\sigma_t\, d)$

The distance at which $\tau = \xi$ is $d = -\log(\xi) / \sigma_t$

Choosing d using this formula from uniform random ξ ($p(\xi) = 1$)leads to the pdf $p(d) = \sigma_t\, \tau(d)$.

Using this pdf leads to the estimator:

$$g = \frac{f}{p} = \frac{\tau(x,x')\ \epsilon(x',\omega)}{\sigma_a\ \tau(x,x')} = \frac{\epsilon(x',\omega)}{\sigma_a}$$

4. Emission + (inhomogeneous) absorption

We can approach this using the same importance sampling scheme as above, but it is no longer so simple to compute the attenuation. Instead of just computing an exponential, we have to numerically integrate the attenuation coefficient along the ray.

The disappointing thing is that we can't just plug in a Monte Carlo estimator for the attenuation integral. If it was true that $E\{\exp(X)\} = \exp(E\{X\})$ then we'd be in good shape—but sadly it is not. This means that the only thing we can do is put in a sufficiently accurate numerical estimate of the integral: we can't count on statistical averaging to let us use a very noisy estimate.

Usually the attenuation integral is computed using some very simple quadrature rule—for example, the midpoint rule—on a regularly spaced set of samples along the ray. Establishing a suitable spacing requires knowing something about how fast the attenuation coefficient can vary in the volume.

So assuming we have some scheme for computing the integral, we can make an estimator from it:

$$g = \frac{\tau(x,x')\ \epsilon(x',\omega)}{p(x')}$$

but again, it is crucial to importance sample τ, which no longer can be done using the simple computation for homogeneous media. However, the generalization is simple. Again, the idea is to compute the distance at which $\tau = \xi$ (for a uniform random ξ), which can be done by modifying the numerical integrator so that it marches along the ray from the starting point, accumulating attenuation, until it just crosses ξ, then interpolates to find the sample point. The pdf that results from this is

$$p(t) = \cancelto{1}{p(\xi)} \left|\frac{d\xi}{dt}\right|$$

$$\frac{d\tau}{dt} = \frac{d}{dt} \exp\left(-\int_0^t \sigma_t(x(t'))\,dt'\right)$$

$$= -\exp\left(-\int_0^t \sigma_t(x(t'))\,dt'\right) \sigma_t(x(t))$$

$$= \tau(x,x')\ \sigma_t(x')$$

(Note here that $d\tau/dt = d\xi/dt$.) If we use this pdf for importance sampling the estimator becomes:

$$g = \frac{\tau(x,x')\ \epsilon(x',\omega)}{p(x')} = \frac{\epsilon(x',\omega)}{\sigma_t(x')}$$

This all leads to the following algorithm for sampling an inhomogeneous medium:

1. Choose $\xi \in [0, 1]$ and find the point x' on the ray for which $\tau(x, x') = \xi$.

2. Find the emission $\varepsilon(x', \omega)$ and compute the estimator $\varepsilon(x', \omega) / \sigma_t(x')$.

5. Introducing the scattering term

We now know how to handle absorption and emission in the general case. The remaining issue is how to compute scattering. As we've observed before, outscattering behaves exactly like absorption, as far as attenuation is concerned. Similarly, inscattering behaves exactly like emission, as far as the integral along the ray is concerned. So we can use the method from the previous section but modify the emission by adding a Monte Carlo estimator for the scattering integral.

Estimating the scattering integral in a volume is just like estimating the reflection integral at a surface. It is an integral over the sphere, and we will choose directions to sample, thinking of them as directions on the sphere.

$$\epsilon_s(x', \omega) = \sigma_s(x') \int_{S^2} f_p(x', \omega, \omega') \, L(x', \omega') \, d\omega'$$

The first thing to try is importance sampling by the phase function—fortunately this is easy to do for Henyey-Greenstein, the only phase function model you're likely to encounter. Remember the phase function is normalized to be a probability distribution already, so in this case $p(\omega') = f_p(x', \omega, \omega')$. Then the estimator is

$$g(\omega') = \frac{f}{p} = \frac{\sigma_s(x') \, f_p(x', \omega, \omega') \, L(x', \omega')}{p(\omega')}$$

$$g(\omega') = \sigma_s(x') \, L(x', \omega')$$

So all we need to do is trace a recursive ray to estimate $L(x', \omega')$ and the process becomes:

1. Choose ξ in $[0, 1]$ and find the point x' on the ray for which $\tau(x, x') = \xi$.

2. Choose ω' according to the phase function.

3. Recursively estimate $L(x', \omega')$ and compute the estimator $(\varepsilon(x', \omega) + \sigma_s(x') \, L(x', \omega')) / \sigma_t(x')$.

Doing this leads to a path tracing algorithm very similar to a brute-force surface path tracer. Adding luminaire sampling, multiple importance sampling, and Russian Roulette all requires straightforward generalizations of the methods used for surfaces.