Cornell CS667 **Lecture : Volumetric diffusion** 4 April 2006

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1 Introduction

From the Light Transport Equation(LTE)

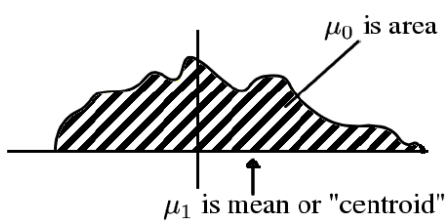
$$(\omega \cdot \nabla)L(x,\omega) = -\sigma_t L(t,\omega) + \sigma_s \int P(x,\omega,\omega')L(x,\omega')d\omega' + Q(x,\omega) \tag{1}$$

We want to find the moment.

1.1 Defining Moments

First, lets define moment. n^{th} moment of $f(x)$ is $\int x^n f(x)dx$.

On real line, the zeroth and first moment are the area under the function and the mean of the distribution.



We are intersted in functions that are distribution over the sphere of direction – that is, function $f(\omega)$ where ω is a unit vector. The moment are a little more complicated because the argument is a vector. But it's not so bad if we just need that first couple.

0^{th} moment - still the integral of the whole thing. For the radiance function at a point in a volume, this is the fluence or scalar irradiance:

$$\phi(x) = \int_{4\pi} L(x,\omega)d\omega \tag{2}$$

1^{st} moment - in some sense, this is the average direction of light flow. The three components of the first moment are just the integrals weighted by $\omega_x, \omega_y, \omega_t$ (the components of n):

$$\vec{E}(x) = \int_{4\pi} \omega L(x,\omega) d\omega = \int_{4\pi} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} L(x,\omega) d\omega = \begin{bmatrix} \int_{4\pi} \omega_x L(x,\omega) d\omega \\ \int_{4\pi} \omega_y L(x,\omega) d\omega \\ \int_{4\pi} \omega_z L(x,\omega) d\omega \end{bmatrix} \tag{3}$$

Lemma: Note that the first moment of a constant function of ω is zero:

$$\begin{aligned} \int_{4\pi} \omega_x C d\omega = C \int_{4\pi} \omega_x d\omega = C \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\theta)(\cos(\theta) d\theta d\phi) &= 2\pi C \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\theta)\cos(\theta) d\theta \\ &= 2\pi C \left[\sin(\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0 \end{aligned}$$

... and similar for other components

A briefer way of writing this is : $\int_{4\pi} \omega d\omega = \vec{0}$

Corollary: The 0^{th} moment of a linear functional ($f(\omega) = a \cdot \omega$) is zero.

This expands to a sum of 3 cases at the previous observation: $\int_{4\pi}(a_x\omega_x + a_y\omega_y + a_z\omega_z) d\omega = a_x \int_{4\pi} \omega_x d\omega + a_y \int_{4\pi} \omega_y d\omega + a_z \int_{4\pi} \omega_z d\omega$

Lemma: The first moment of a linear functional ($f(\omega) = a \cdot \omega$) is $\frac{4\pi}{3} a$.

This is $\int_{4\pi} \omega(a \cdot \omega) d\omega$. $(a \cdot \omega)$ is a scalar. $\omega(a \cdot \omega)$ is a vector.Components are

$$\int_{4\pi} (\omega_i(a_1\omega_1 + a_2\omega_2 + a_3\omega_3) d\omega = \sum_j a_j \int_{4\pi} \omega_i \omega_j d\omega \tag{4}$$

When $i \neq j$: $\omega_i\omega_j$ is antisymmetric across the $w_i = 0$ plane $\Rightarrow \int$ is zero.

For $i = j$: have $\int {\omega_i}^2 d\omega$. But

$$\begin{aligned} \int_{4\pi} ({\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2) d\omega &= \int_{4\pi} 1 d\omega = 4\pi \\ &= \int_{4\pi} {\omega_1}^2 d\omega + \int {\omega_2}^2 d\omega + \int {\omega_3}^2 d\omega; \text{ by symmetry} \\ &= 3 \int_{4\pi} {\omega_i}^2 d\omega \end{aligned}$$

So $\int_{4\pi} {\omega_i}^2 d\omega = \frac{4\pi}{3}$. So each component is $\frac{4\pi}{3} a_i$, result is $\frac{4\pi}{3} a$.

Lemma: The 0^{th} moment of a quadratic form $f(\omega) = \omega^T A \omega = \omega \cdot (A\omega)$ is $\frac{4\pi}{3} Tr(A)$ (where $Tr(A) = \sum_i a_{ii}$ (sum of diangonal elements)). This is another instance of the above reasoning. $\omega^T A \omega$ is $\sum_{ij} \omega_i a_{ij} \omega_j$. So the integral expands to

$$\sum_{ij} a_{ij} \int \omega_i \omega_j d\omega$$

,

$$\int \omega_i \omega_j d\omega \rightarrow 0 \text{ for } i \neq j, \frac{4\pi}{3} \text{ for } i = j$$

$$= \frac{4\pi}{3}\sum_i a_{ii} = \frac{4\pi}{3}Tr(A)$$

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1.2 Moments Summary

To summarize:

	μ_0	μ_1
c	$4\pi c$	0
$a \cdot \omega$	0	$\frac{4\pi}{3}a$
$\omega^T A\omega$	$\frac{4\pi}{3}Tr(A)$	0 (symmetry argument)

One more lemma, about spatial derivatives. The directional derivative of a linear functional that varies spatially: $(v \cdot \nabla)(a(x) \cdot \omega) = v \cdot (\nabla a(x) \cdot \omega) = \omega^T(\nabla a(x))v$

$$\begin{aligned}(v \cdot \nabla)(a(x) \cdot \omega) &= \sum_j v_j \frac{\partial}{\partial x_j}(a(x) \cdot \omega) \\ &= \sum_j v_j \frac{\partial}{\partial x_j}(\sum_i a_i(x)\omega_i) \\ &= \sum_{ij} \omega_i \frac{\partial a_i}{\partial x_j}(x) v_j \\ &= \sum_{ij} \omega_i (\nabla a)_{ij} v_j\end{aligned}$$

1.3 Moment of phase funtion

If we fix one direction, the phase function is a function of direction.
$f(w) = p(\omega_0, \omega) \leftarrow$ depends only on $\omega_0 \cdot \omega$ — i.e. rotationally symmetric about ω_0.

0^{th} moment: $\int_{4\pi} p(\omega_0, \omega)d\omega = 1$ (it's a probablity distribution)
1^{st} moment: $\int_{4\pi} p(\omega_0, \omega)\omega d\omega$. This is the average direction of scattering, in some sense.

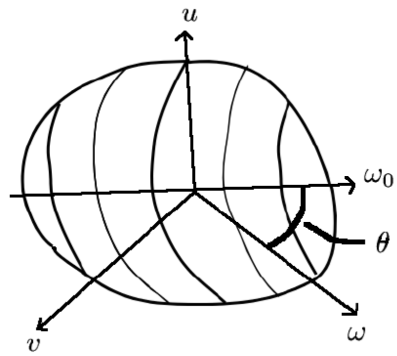
By symmetry this has got to point in the direction $\pm\omega_0$. So express ω in a coordinate system (u, v, w_0) where u and v are chosen arbitrarily to complete the ONB. The three components are then

$$\int p(\omega_0, \omega)(\omega_0 \cdot \omega)d\omega \rightarrow \text{ this is t he familar g}$$

$$\int p(\omega_0, \omega)(u * \omega)d\omega = 0)\} \text{ by symmetry. + and - hemisphere cancel}$$

$$\int p(\omega_0, \omega)(v * \omega)d\omega = 0)\} \text{ by symmetry. + and - hemisphere cancel}$$

So the moment $\mu_1(p(\omega_0, \omega)) = g\omega_0$



2 Approximation of the volumetric Light Transport Equation

Now to get to the main point. From the volume LTE eq(1), we make the assumption that radiance is directionally smooth enought to be approximated by a 1^{st} order function in ω:

$$L(x,\omega) = \frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega \cdot E(x) \leftarrow \quad \begin{array}{l} \mu_0(\frac{1}{4\pi}\phi(x)) = \phi(x), \mu_0(\frac{3}{4\pi}\omega \cdot E(x)) = 0 \\ \mu_1(\frac{1}{4\pi}\phi(x)) = 0, \mu_1(\frac{3}{4\pi}\omega \cdot E(x)) = E(x) \\ \text{so agrees up to } 1^{st} \text{ order with } L(x,\omega) \end{array}$$

Now substitution

$$(\omega\cdot\nabla)(\frac{1}{4\pi}\phi(x)+\frac{3}{4\pi}\omega\cdot E(x))+\sigma_t(\frac{1}{4\pi}\phi(x)+\frac{3}{4\pi}\omega\cdot E(x)) = \sigma_s(x)\int_{4\pi} p(x,\omega,\omega')(\frac{1}{4\pi}\phi(x)+\frac{3}{4\pi}\omega'\cdot E(x))d\omega'+Q(x,\omega) \tag{5}$$

Now that we have restricted the radiance function to this particular form, we can no longer expect it to solve the LTE exactly. Instead we will loook for agreement up to first order: The 0^{th} and 1^{st} moments of the two sides should match.

What's the 0^{th} moment at this eqn?

$$\text{LHS:} \quad \begin{array}{ccccccc} \frac{1}{4\pi}\int_{4\pi}\omega\cdot\nabla\phi(x)d\omega & + & \frac{3}{4\pi}\int_{4\pi}(\omega\cdot\nabla)(E(x)\cdot\omega)d\omega & + & \frac{\sigma_t}{4\pi}\int_{4\pi}\phi(x)d\omega & + & \frac{3\sigma_t}{4\pi}\int\omega\cdot E(x)d\omega \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \int\omega\cdot a = 0 & & (\omega\cdot\nabla)(\omega\cdot a(x)) & & 4\pi\phi(x) & & \int\omega\cdot a = 0 \\ & & \frac{3}{4\pi}\int_{4\pi}(\omega^T(\nabla E(x))\omega)d\omega & & \sigma_t\phi(x) & & \\ & & \int\omega^T A\omega = \frac{4\pi}{3}Tr(A) & & & & \\ & & Tr(\nabla E) = \nabla\cdot E & & & & \end{array}$$

$\mu_0(LHS) = \nabla\cdot E(x) + \sigma_t\phi(x)$

$$\frac{\sigma_s(x)}{4\pi}\int_{4\pi}\int_{4\pi} \underset{\substack{\uparrow \\ \phi(x) \text{ is const}}}{p(x,\omega,\omega')\phi(x)d\omega' d\omega} \quad + \quad \frac{3\sigma_s(x)}{4\pi}\int_{4\pi}\int_{4\pi} \underset{\substack{\uparrow \\ E(x) \text{ is const}}}{p(x,\omega,\omega')\omega'\cdot E(x)d\omega' d\omega} \quad + \quad Q_0(x)$$

RHS:

$$\begin{array}{cc} \int p(\omega_0,\omega)d\omega = 1 & \int p(\omega_0,\omega)d\omega = g\omega \\ \sigma_s(x)\phi(x) & \frac{3\sigma_s(x)}{4\pi}\int_{4\pi} g\omega\cdot E(x)d\omega \\ & \int \omega\cdot a = 0 \end{array}$$

$\mu_0(RHS) = \sigma_s(x)\phi(x) + Q_0$

So in the end, the 0^{th} moment reads: $\nabla\cdot E(x) + \sigma_t(x)\phi(x) = \sigma_s(x)\phi(x) + Q_0(x)$ or $\nabla\cdot E(x) = (\sigma_s - \sigma_t)(x)\phi(x) + Q_0(x) = \sigma_a(x)\phi(x) + Q_0(x)$

This says something not too surprising about the flow of light: The vector irradiance is something that tells us about the net flow of power across a surface, and the divergence of that says how much it's flowing into or out of an area. Power flow out of areas where there are sources, and it flows into (disappears from) areas that have absorption.

So far this is just one statement about 4 functions(ϕ, E_1, E_2, E_3). To get more constraints we look at the 1^{st} order.

LHS:

$$\underset{\substack{\uparrow \\ \mu_1(\omega\cdot a) = \frac{4\pi}{3}a}}{\frac{1}{4\pi}\int \omega(\omega\cdot\nabla\phi(x))d\omega} \quad + \quad \underset{\substack{\uparrow \\ \mu_1(\omega^T A\omega) = 0}}{\frac{3}{4\pi}\int \omega(\omega^T\nabla E(x)\omega)d\omega} \quad + \quad \underset{\substack{\uparrow \\ \mu_1(c) = 0}}{\frac{\sigma_t}{4\pi}\int \omega\phi(x)d\omega} \quad + \quad \underset{\substack{\uparrow \\ \mu_1(\omega\cdot a) = \frac{4\pi}{3}a}}{\frac{3\sigma_t}{4\pi}\int \omega(\omega\cdot E(x))d\omega}$$

$\mu_1(LHS) = \frac{1}{3}\nabla\phi(x) + \sigma_t E(x)$

$$\frac{\sigma_s(x)}{4\pi}\int \omega\int p(x,\omega,\omega')\phi(x)d\omega' d\omega \quad + \quad \frac{3\sigma_s(x)}{4\pi}\int \omega\int p(\omega,\omega')\omega'\cdot E(x)d\omega' d\omega \quad + \quad Q_1(x)$$

RHS:

$$\begin{array}{cc} \phi(x) \text{ is const} & E(x) \text{ is const} \\ \int p(\omega,\omega')d\omega = 1 & \int p(\omega,\omega')\omega'\cdot E(x)d\omega' = g\omega\cdot E(x) \\ \mu_1(c) = 0 & \mu_1(a\cdot\omega) = \frac{4\pi}{3}a \\ & \frac{4\pi}{3}gE(x) \\ & \sigma_s(x)gE(x) \end{array}$$

$\mu_1(RHS) = \sigma_s(x)gE(x) + Q_1(x)$

$$\begin{aligned} \text{So } 1^{st} \text{ order } \rightarrow \quad \tfrac{1}{3}\nabla\phi(x) + \sigma_t E(x) &= \sigma_s(x)gE(x) + Q_1(x) \\ \tfrac{1}{3}\nabla\phi(x) &= (g\sigma_s(x) - \sigma_t(x))E(x) + Q_1(x) \\ &= -(\sigma_a + (1-g)\sigma_s)E(x) + Q_1(x) \\ \nabla\phi(x) &= -3\sigma_t' E(x) + 3Q_1(x) \end{aligned}$$

$(1-g)\sigma_s \rightarrow$ reduced scattering coefficent: effective scattering for diffusion; narrower peaks has same effect as less scattering.

If sources are isotropic, $Q_1(x) = 0$. This makes for a simpler 1^{st} case:

$\nabla\phi(x) = -3\phi_t'(x)E(x)$ (1^{st} order)
$E(x) = -\frac{1}{3\sigma_t'(x)}\nabla\phi(x)$ or just $E = \frac{-1}{3\sigma_t'}\nabla\phi$
substitute into zero-order equation

$\nabla \cdot E = -\sigma_a\phi + Q_0$
$\nabla \cdot (\frac{-1}{3\sigma_t'}\nabla\phi) = -\sigma_a\phi + Q_0$
$\frac{-1}{3\sigma_t'}\nabla^2\phi = -\sigma_a\phi = Q_0$
$\nabla^2\phi \leftarrow$ Laplacian of ϕ - scalar 2^{nd} derivative $\sum_i \frac{\partial^2\phi}{\partial x_i \partial x_i}$

or, in more classic form

$$\boxed{\nabla^2\phi = 3\sigma_a\sigma_t'\phi - 3\sigma_t'Q_0} \tag{6}$$

If we keep Q_1,

$$E = -\frac{1}{3\sigma_t'}\nabla\phi + \frac{1}{\sigma_t'}Q_1$$
$$-\frac{1}{3\sigma_t'}\nabla^2\phi + \frac{1}{\sigma_t'}\nabla \cdot Q_1 = -\sigma_a\phi + Q_0$$
$$D\nabla^2\phi = \sigma_a\phi - Q_0 + 3D\nabla \cdot Q_1$$

where $D = \frac{1}{3\sigma_t'}$. This is the form used by Jensen et. al 01.