CS667 **Rigid Bodies** 25 April 2006

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1 Introduction

Simulating a rigid body system is similar to simulating particles except for one key difference: a rigid body has an additional degree of freedom—they can rotate. Given outside forces acting on points on the rigid body, our goal is deriving the equations for the rigid body's motion.

The convention in this lecture is to describe the rigid body as a rigidly connected set of point masses, but this concept transfers easily to continuous mass distribution.

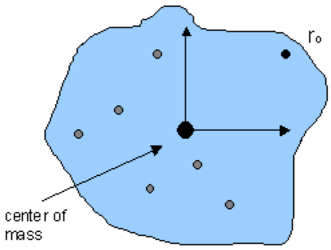
A good reference for all this material is David Baraff's Siggraph 1997 course notes on Rigid Body Dynamics (available at *http://www.cs.cmu.edu/ baraff/sigcourse/*).

Step one will be describing motion of the rigid body system (**kinematics**).

Step two will be the equations that govern the motion (**dynamics**).

2 Kinematics

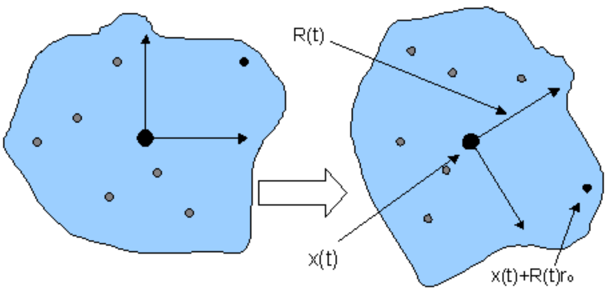
Here is a rigid body:



The rigid body consists of mass points m_i at locations $r_{o,i}$ in the body's coordinate system.

Take body coordinates as always centered around the center of mass (this will simplify our arithmetic).

At time t, this rigid body has some rigid motion applied to it:



Specifically, the motion has the following form:

$$\text{Position: } r(t) = x(t) + R(t)r_o$$
$$\text{Velocity: } \dot{r}(t) = \dot{x}(t) + \dot{R}(t)r_o$$

We know $\dot{x}(t)$ is the velocity $v(t)$. What does $\dot{R}(t)r_o$ look like? Suppose we create a tiny rotation from $t \to t + \delta t$. This produces the following rotation:

$$R(t + \delta t) = \delta R \cdot R(t)$$

How does this rotation δR, with axis ω and an angle $\delta\theta$, effect an arbitrary vector v?

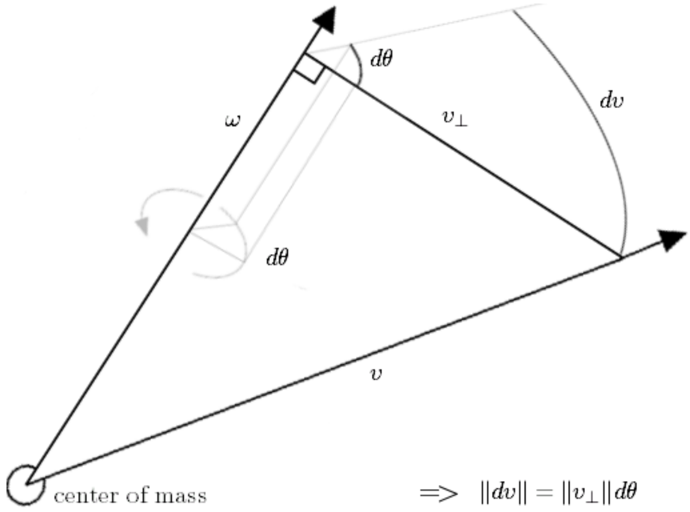


Figure 1: The rotation δR with axis of rotation ω and (small) angle $d\theta$.

From Figure 1, we know (1) $dv = \|v_\perp\| d\theta$ and (2) dv is perpendicular to ω and v. So we can represent dv as $dv = \omega \times v$—which is valid because of (2)—with length $\|\omega\|\|v_\perp\|$. Because of (1), we can then encode $d\theta$ as $\|\omega\|$.

In this form, ω is called the angular velocity.

- The direction of ω is the instantaneous rotation axis.
- The magnitude of ω is instantaneous rotation rate $\frac{\delta\theta}{\delta t}$.

The velocity of any point r rotation under $R(t)$ is

$$\dot{r}(t) = \omega(t) \times r(t)$$

This applies equally well to the columns of R, so

$$\dot{R}(t) = [\omega \times u_1(t), \omega \times u_2(t), \omega \times u_3(t)] = \omega \times R(t)$$

So the velocity at some point fixed to a moving frame $x(t) + R(t)r_o$ is

$$\begin{aligned}\dot{r}(t) &= \dot{x}(t) + \dot{R}(t)r_o \\ &= v(t) + (\omega(t) \times R(t))v_o \\ &= v(t) + \omega(t) \times r'(t) \text{ where } r'(t) = R(t)r_o = r(t) - x(t)\end{aligned}$$

Thus kinematics can be summarized in terms of variables, velocities, and transformations to world space:

- Variables are $x(t)$, $r(t)$.
- Velocities:

$$\underbrace{V(t)}_{\text{linear velocity}} = \dot{x}(t).$$

$$\underbrace{\omega(t)}_{\text{angular velocity}} \quad \text{where } \dot{R}(t) = \omega(t) \times R(t).$$

- Transformations to world space:

  World space position of point r_o is $r(t) = x(t) + \underbrace{R(t)r_o}_{r'(t)}$.

  World space velocity is $\dot{r}(t) = v(t) + \omega \times r'(t)$.

3 Dynamics

Now we look at dynamics, deriving equations for x, V, R, ω over time.

We'll find these equations in a roundabout fashion. Let us discuss conservation of energy first, to establish conserved quantities that will make better state variables.

Suppose our rigid body in flight, tumbling through space. Its kinetic energy T will be conserved (in the absence of any potential energy).

$$\begin{aligned} T &= \sum \frac{1}{2} m_i (v_i \cdot v_i) = \sum \frac{1}{2} m_i (\dot{x}(t) + \omega \times r'_i(t)) \\ &= \sum \frac{1}{2} m_i (\dot{x}(t)^2 + \dot{x}(t) \cdot \omega \times r'_i(t) + (\omega \times r'_i(t))^2) \\ &= \frac{1}{2} M V^2 + \dot{x} \cdot \omega \times \underbrace{\sum m_i \cdot r'_i}_{\text{center of mass } = 0} + \frac{1}{2} \sum m_i \|\omega \times r'_i\|^2 \end{aligned}$$

The center of mass is zero because we are operating in a center of mass coordinate system. Now we note that

$$\begin{aligned} \|\omega \times v\|^2 &= \|\omega\|^2 \|v\|^2 \sin^2(\theta) \\ &= \|\omega\|^2 \|v\|^2 (1 - \cos^2(\theta)) \\ &= (\omega \cdot \omega)(v \cdot v) - (\omega \cdot v)^2 \\ &= \omega^{\mathrm{T}} (v^{\mathrm{T}} v) \omega - \omega^{\mathrm{T}} (v v^{\mathrm{T}}) \omega \\ &= \omega^{\mathrm{T}} (v^{\mathrm{T}} v \; \mathbf{1} - v v^{\mathrm{T}}) \omega && \text{where } \mathbf{1} \text{ is the 3x3 identity matrix} \end{aligned}$$

Plug in the above identity into our equation for T:

$$\begin{aligned} &= \frac{1}{2} M V^2 + \frac{1}{2} \sum m_i ((\omega \cdot \omega)(r'_i \cdot r'_i) - (\omega \cdot r'_i)^2) \\ &\quad \frac{1}{2} \sum m_i (\omega^{\mathrm{T}} ((r'_i)^{\mathrm{T}} r'_i) \omega - \omega^{\mathrm{T}} (r'_i (r'_i)^{\mathrm{T}}) \omega) \\ &\quad \frac{1}{2} \omega^{\mathrm{T}} \underbrace{\left[\sum m_i (((r'_i)^{\mathrm{T}} r'_i) \mathbf{1} - (r'_i (r'_i)^{\mathrm{T}})) \right]}_{\text{inertia tensor } I \text{ (a symmetric 3x3 matrix)}} \omega \end{aligned}$$

With the above equation, we can see that kinetic energy T can be broken into two components:

$$T = \underbrace{\frac{1}{2}MV^2}_{\text{kinetic energy of translational motion}} + \underbrace{\frac{1}{2}\omega^{\mathrm{T}} I\omega}_{\text{kinetic energy of rotational motion}}$$

Some interesting points about T:

- Translational kinetic energy is the same as if there was no rotation whatsoever.
- Rotational kinetic energy is related quadratically to ω (like translational kinetic energy).
- Kinetic energy of a particular angular velocity *depends on the direction.*

Now we know the derivative of this energy is zero (assuming no input or output energy).

$$\frac{\delta T}{\delta t} = 0 = \underbrace{MV}_{\text{linear momentum } P} + \underbrace{I\omega}_{\text{angular momentum } L}$$

where linear momentum $P = \sum P_i$ and angular momentum $L = \sum r'_i \times P_i$ (moment of momentum). Because these two momentum terms are conserved quantities – at least their sum is – these are usually independent variables of choice (instead of V and ω).

One remark on I: I depends on R (the rotation of the rigid body) since it is calculated from the r'_i. But it doesn't really–it's just a change of coordinates:

$$\begin{aligned} I &= \sum m_i((r_i'^{\mathrm{T}} r_i')\mathbf{1} - (r_i' r_i'^{\mathrm{T}})) \\ &= \sum m_i(((R(t)r_{o,i})^{\mathrm{T}}(R(t)r_{o,i}))\mathbf{1} - ((R(t)r_{o,i})(R(t)r_{o,i})^{\mathrm{T}})) && \text{since } r'_i = R(t)r_{o,i} \\ &= \sum m_i((r_{o,i}^{\mathrm{T}} R(t)^{\mathrm{T}} R(t) r_{o,i})\mathbf{1} - R(t)r_{o,i}r_{o,i}^{\mathrm{T}} R(t)^{\mathrm{T}}) && (1) \end{aligned}$$

Note that

$$r_{o,i}^{\mathrm{T}} R(t)^{\mathrm{T}} R(t) r_{o,i} = r_{o,i}^{\mathrm{T}}(\mathbf{1})r_{o,i} = r_{o,i}^{\mathrm{T}} r_{o,i}\mathbf{1} = r_{o,i}^{\mathrm{T}} r_{o,i} R(t)R(t)^{\mathrm{T}} \qquad (2)$$

Plugging in (2) into (1):

$$\begin{aligned} &= \sum m_i((r_{o,i}^{\mathrm{T}} r_{o,i})R(t)R(t)^{\mathrm{T}}\mathbf{1} - R(t)r_{o,i}r_{o,i}^{\mathrm{T}} R(t)^{\mathrm{T}}) \\ &= \sum m_i(R(t)(r_{o,i}^{\mathrm{T}} r_{o,i})R(t)^{\mathrm{T}}\mathbf{1} - R(t)r_{o,i}r_{o,i}^{\mathrm{T}} R(t)^{\mathrm{T}}) && \text{since } r_{o,i}^{\mathrm{T}} r_{o,i} \text{ is a scalar} \\ &= R(t) \underbrace{\left[\sum m_i((r_{o,i}^{\mathrm{T}} r_{o,i})\mathbf{1} - r_{o,i}r_{o,i}^{\mathrm{T}})\right]}_{\text{inertia tensor in body coordinates } (I_{\text{body}} \text{ in Baraff})} R(t)^{\mathrm{T}} \end{aligned}$$

Momentum and angular momentum stay constant unless there are external forces. Suppose there is a force f_i applied to a particle i. This force includes internal forces, but we know those have to sum to zero in their effect on momentum.

The effect on linear momentum is easy to compute:

$$\begin{aligned} \dot{p}_i &= f_i && \text{and} \\ \dot{P} &= \sum \dot{p}_i = F \\ &= \text{total sum of external forces (internal forces cancel out)} \end{aligned}$$

We can also find the effect on angular momentum:

$$\dot{L} = \sum r_i' \times \dot{p}_i = \sum r_i' \times f_i = \tau \qquad \text{where } \tau \text{ is the } \textit{torque}\text{, the moment of force}$$

Recall that we are ignoring internal forces since they will not change the angular momentum. That is because (a) angular momentum is conserved (in the absence of external forces) and (b) the internal forces form equal and opposite pairs.

Thus dynamics can be summarized as the following equations of motion:

- State variables are $x(t)$, $R(t)$.
- Velocities are $v(t)$, $\omega(t)$.
- Equations of motion:

$$\dot{x} = v$$
$$m\dot{v} = \dot{P} = F$$
$$\dot{R} = \omega \times R$$
$$I\dot{\omega} = \dot{L} = \tau$$