CS667 **Lecture 17: Subsurface Light Diffusion** 14 April 2005

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1 Mathematical Preliminaries

Our discussion of Subsurface Light Diffusion relies heavily on two mathematical concepts: moments and derivatives. Therefore, before delving into the main topic of the lecture we will first review this essential material. The familiar reader can skip this section, with the caveat that at several times in our future derivations we will reference lemmas that we derive here.

1.1 Moments

The n^{th} moment of a function $f(x)$ is defined to be:

$$\mu_n(f) = \int x^n f(x) dx$$

So, for a one-dimensional function over the real line, the zeroth moment represents the area under the curve. If this function is a probability distribution (a pdf), then the first moment corresponds to the mean of the distribution.

In the context of light transport we will be considering functions that are distributions over the sphere of possible directions (i.e. $f(\omega)$ where ω is a unit vector). The moments of such a function are slightly more complicated, but we will only be using the first two:

$$\mu_0(f(\omega)) = \int_{4\pi} f(\omega) d\omega$$

$$\mu_1(f(\omega)) = \int_{4\pi} \omega f(\omega) d\omega$$

In terms of light transport, the 0^{th} moment of the radiance function at a point x in space, $L(x, \omega)$, is the fluence (or scalar irradiance), $\phi(x)$:

$$\mu_0(L(x,\omega)) = \int_{4\pi} L(x,\omega) d\omega = \phi(x)$$

The 1^{st} moment of a function $f(\omega)$ is a vector. We can write it in terms of its components:

$$\mu_1(f(\omega)) = \int_{4\pi} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} f(\omega) d\omega = \begin{bmatrix} \int_{4\pi} \omega_x f(\omega) d\omega \\ \int_{4\pi} \omega_y f(\omega) d\omega \\ \int_{4\pi} \omega_z f(\omega) d\omega \end{bmatrix} \quad (1)$$

Intuitively, the first moment can be thought of as pointing in the average direction of the function. For light transport this is the average direction of light flow, called the vector irradiance $\vec{E}(x)$.

We will now derive some lemmas about the moments of different classes of functions $f(\omega)$ over the unit sphere.

1.1.1 Constant Functions

For a constant function $f(\omega) = C$:

$$\mu_0(C) = \int_{4\pi} C d\omega = 4\pi C \tag{2}$$

$$\mu_1(C) = \int_{4\pi} C\omega d\omega = \vec{0} \tag{3}$$

(2) is a direct result of solving the integral. Intuitively, the reason for (3) is that all pairs of vectors pointing in opposite directions have equal magnitude (C) and thus cancel. More concretely, recalling that we can treat $\mu_1(f(\omega))$ as three separate integrals (as shown in (1)), we examine one of the components of the resulting vector:

$$\int_{4\pi} C\omega_i d\omega = C \int_{4\pi} \omega_i d\omega = C \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \cos\theta d\theta d\phi = 2\pi C \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \cos\theta d\theta = 2\pi C [\sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0 \tag{4}$$

The spherical coordinates over which we integrate are aligned such that the component in which we are interested varies only with θ. This can be accomplished by letting θ measure the distance from this component and ϕ the rotation around this component. From the integral above we see see that all components are 0 and thus the resulting vector is $\vec{0}$. Briefly stated, we have shown that $\int_{4\pi} \omega d\omega = \vec{0}$.

1.1.2 Linear Functions

For a linear function: $f(\omega) = \vec{a} \cdot \omega$

$$\mu_0(\vec{a} \cdot \omega) = \int_{4\pi} (\vec{a} \cdot \omega) d\omega = \int_{4\pi} (a_x\omega_x + a_y\omega_y + a_z\omega_z) d\omega$$

This integral can be divided into three integrals of the form $\int_{4\pi} a_i\omega_i d\omega$ which, as we showed in (4), are each equal to 0. Thus the entire integral is 0 and we conclude:

$$\mu_0(\vec{a} \cdot \omega) = 0 \tag{5}$$

We can write the first moment as:

$$\mu_1(\vec{a} \cdot \omega) = \int_{4\pi} \omega(\vec{a} \cdot \omega) d\omega$$

Again, we examine an individual component of the resulting vector:

$$(\mu_1(\vec{a} \cdot \omega))_i = \int_{4\pi} \omega_i (\sum_j a_j\omega_j) d\omega = \sum_j a_j \int_{4\pi} \omega_i\omega_j d\omega$$

To evaluate this integral, notice that if $i \neq j$ then $\omega_i\omega_j$ is antisymmetric across the $\omega_i = 0$ plane and thus $\int_{4\pi} \omega_i\omega_j d\omega = 0$. Otherwise, if $i = j$ we need to evaluate $\int_{4\pi} \omega_i^2 d\omega$. We know that $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$ since ω is a vector on the unit sphere. Therefore,

$$\int (\omega_1^2 + \omega_2^2 + \omega_3^2) d\omega = 4\pi$$

$$\int \omega_1^2 d\omega + \int \omega_2^2 d\omega + \int \omega_3^2 d\omega = 4\pi$$

Since by symmetry all these components must be equal, we conclude that $\int \omega_i^2 d\omega = \frac{4\pi}{3}$. Continuing our reasoning above,

$$(\mu_1(\vec{a}\cdot\omega))_i = \sum_j a_j \int_{4\pi} \omega_i\omega_j d\omega = \sum_j a_j \delta_{ij}\frac{4\pi}{3} = \frac{4\pi}{3}a_i$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & i \neq j \end{cases}$$

Since this is true for each component:

$$\mu_1(\vec{a}\cdot\omega) = \frac{4\pi}{3}\vec{a} \tag{6}$$

1.2 Quadratic Functions

For a quadratic function: $f(\omega) = \omega^T A\omega = \omega \cdot (A\omega) = \sum_{i,j} \omega_i a_{ij}\omega_j$

$$\mu_0(\omega^T A\omega) = \frac{4\pi}{3}Tr(A) \tag{7}$$

where $Tr(A) = \sum_i a_{ii}$ (the trace of the matrix A—the sum of its diagonal entries). This follows from the same reasoning used to derive the first moment of a linear function. We can expand $\int \omega^T A\omega d\omega$ into the sum $\sum_{i,j} a_{ij} \int \omega_i\omega_j d\omega$. From the derivation above we know $\int \omega_i\omega_j d\omega$ will be 0 if $i \neq j$ and $\frac{4\pi}{3}$ if $i = j$. Thus we can remove all terms where $i \neq j$ from the sum, leaving: $\sum_i a_{ii} \int \omega_i\omega_i d\omega = \frac{4\pi}{3}\sum_i a_{ii} = \frac{4\pi}{3}Tr(A)$. Note that if A is the identity matrix then $\mu_0(f(\omega)) = \int \omega^T\omega d\omega = 4\pi$.

And the first moment of a quadratic function:

$$\mu_1(\omega^T A\omega) = \vec{0} \tag{8}$$

This fact follows from a symmetry argument similar to the one used for the first moment of a constant function.

1.2.1 The Phase Function

In light transport problems the phase function $p(\omega_{in}, \omega_{out})$ describes the scattering behavior of the medium. Specifically, it represents the probability that a scattered photon travelling from a direction ω_{in} will alter its path to travel along the vector ω_{out}.

If we fix one direction of the phase function (set $\omega_{in} = \omega_0$) we get a function of direction: $f(\omega^*) = p(\omega_0, \omega^*)$ which depends only on $\omega_0 \cdot \omega^*$. Thus f is rotationally symmetric about ω_0.

$$\mu_0(f(\omega)) = \int_{4\pi} p(\omega_0, \omega)d\omega = 1 \tag{9}$$

This follows directly from the fact that f is a probability distribution (over the directions in which a photon will scatter).

We can write the first moment as:

$$\mu_1(f(\omega)) = \int_{4\pi} p(\omega_0, \omega)\omega d\omega$$

Since f is rotationally symmetric about ω_0, it follows that the first moment will be a scalar multiple of ω_0. So, to solve this integral it will be helpful to define a new basis $(\vec{u}, \vec{v}, \omega_0)$ where $\vec{u}$ and $\vec{v}$ are chosen arbitrarily to satisfy the orthogonality condition. The three components of the first moment are then:

$$\mu_1(f(\omega)) = \begin{bmatrix} \int p(\omega_0, \omega)(\vec{u} \cdot \omega) d\omega \\ \int p(\omega_0, \omega)(\vec{v} \cdot \omega) d\omega \\ \int p(\omega_0, \omega)(\omega_0 \cdot \omega) d\omega \end{bmatrix}$$

The first two components are zero because of symmetry (the upper and lower hemishperes will cancel). The last component is the familiar g. Thus,

$$\mu_1(p(\omega_0, \omega)) = g\omega_0 \tag{10}$$

1.3 Derivatives

Here we will provide a short review of the multi-dimensional derivatives we will use in the derivation. In all cases we assume $\vec{x}$ is a vector in $\Re^3$.

For a scalar function $f(\vec{x})$, the gradient is defined to be:

$$\nabla f(\vec{x}) = \left[\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \frac{\partial}{\partial x_3} f \right]$$

and the directional derivative (the change in function value at point $\vec{x}$ in direction $\vec{u}$) is defined to be:

$$(\vec{u} \cdot \nabla) f = \vec{u} \cdot \nabla f \tag{11}$$

For the special case of a linear function that varies with position, $f(\vec{x}, \omega) = a(\vec{x}) \cdot \omega$ (where ω is fixed), we can simplify the directional derivative, in either vector or summation notation, as follows:

$$\begin{aligned} (\vec{u} \cdot \nabla) f(\vec{x}, \omega) = (\vec{u} \cdot \nabla)(a(\vec{x}) \cdot \omega) &= \sum_j u_j \frac{\partial}{\partial x_j} (a(\vec{x}) \cdot \omega) \\ \vec{u} \cdot (\nabla a(\vec{x}) \cdot \omega) &= \sum_j u_j \frac{\partial}{\partial x_j} \sum_i a_i(\vec{x}) \omega_i \\ \omega^T (\nabla a(\vec{x})) \vec{u} &= \sum_{i,j} \omega_i \underbrace{\frac{\partial a_i}{\partial x_j}(\vec{x})}_{(\nabla a)_{ij}} v_j \end{aligned} \tag{12}$$

For a vector function $F(\vec{x}) = \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \\ f_3(x_3) \end{bmatrix}$, the derivative is defined to be:

$$\nabla F = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

and the divergence is defined to be:

$$\nabla \cdot F = Tr(\nabla F) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

The divergence can be thought of as measuring flow into and out of an infinitely small volume around the point $\vec{x}$. If $\nabla \cdot F = 0$ the flow in equals the flow out. If $\nabla \cdot F > 0$ the flow out is greater than the flow in (and vice versa for < 0).

Finally, the Laplacian (scalar second derivative) of a vector function is:

$$\nabla^2 F = \sum_i \frac{\partial^2 F}{\partial x_i \partial x_i}$$

2 Brief Refresher on Light Scattering

Recall the rendering equation for volumetric light transport:

$$(\omega \cdot \nabla) L(x, \omega) = -\sigma_t L(x, \omega) + \sigma_s \int_{4\pi} p(\omega, \omega') L(x, \omega') d\sigma(\omega') + Q(x, \omega)$$

Where $\sigma_t = \sigma_s + \sigma_a$ denotes the extinction coefficient, $p(\omega, \omega')$ is the phase function, and $Q(x, \omega)$ is energy from the light source.

Below are some examples of what happens to light as it moves through mediums with various scattering coefficients:

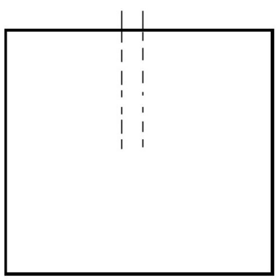

Figure 1: visible structure (low σ_s)

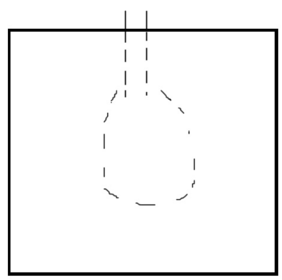

Figure 2: some structure (medium σ_s)

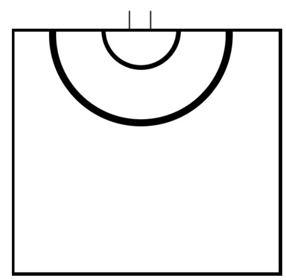

Figure 3: no structure (high σ_s)

The last example, depicted in Figure 3, results from what is essentially isotropic scattering where a ray of light travels a very small distance in any given direction before randomly altering its course. Standard MC-based are too slow for computing light transport in high-scattering media such and simulating materials such as marble, milk, and skin require a different approach.

3 Describing Subsurface Scattering - The BSSRDF

Much of the early work in subsurface scattering came from the medical physics community where diffusion of laser beams in human tissue was of great interest. Jim Kajiya and Jos Stam were among the first to introduce diffusion to computer graphics.

To describe subsurface scattering we need something more powerful than the BRDF (introduced by Nicodemius et al. in *Geometric considerations and nomenclature for reflectance*).

The bidirectional surface scattering reflectance distribution function (BSSRDF) is a generalization of the BRDF.

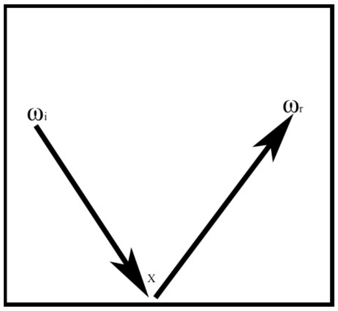

Figure 4: BRDF: $f_r(x, \omega_i, \omega_r)$

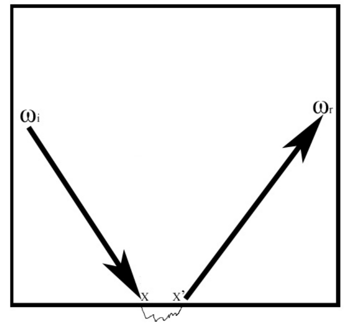

Figure 5: BSSRDF: $S(x, \omega_i, x', \omega_r)$

From the definitions of the BRDF and the BSSRDF, equations for light transport between two points are now:

BRDF:

$$L_r(x_r, \omega_r) = \int_{\mathcal{H}^2} f_r(x_r, \omega_r, \omega_i) d\mu(\omega_i)$$

BSSRDF:

$$L_r(x_r, \omega_r) = \int_{\mathcal{A}} \int_{\mathcal{H}^2} S(x_i, \omega_i, x_r, \omega_r) L_i(x_i, w_i) d\mu(\omega_i) dA(x_i)$$

Where the units of measurement are:

$$S = \frac{dL}{d\Phi} = \frac{\text{reflected radiance}}{\text{incident power}} = \frac{1}{m^2\text{sr}} \qquad \text{and}$$

$$f_r = \frac{dL}{dE} = \frac{\text{reflected radiance}}{\text{irradiance}} = \frac{1}{\text{sr}}$$

4 Simplification of the Light Transport Equation

We are interested in the behavior of light after many scattering events and can make the following assumption: after n scattering events light behaves as if it was traveling through an isotropically-scattering medium.

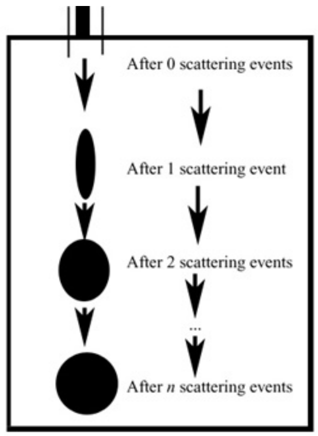

Figure 6: Convolution of the phase function with itself after many scattering events

4.1 An Approximation for $L(x)$

Using this assumption, we can approximate the true value of $L(x, \omega)$ as a linear function of the reflection angle ω:

$$\tilde{L}(x, \omega) = \underbrace{L_0(x)}_{\text{constant component}} + \underbrace{\omega \cdot L_1(x)}_{\text{linear component}}$$

We can now simplify the volume transport equation by substituting in $\tilde{L}$.

Zeroth moment becomes:

$$\text{Fluence } \phi(x) = \int_{4\pi} L d\sigma = \int_{4\pi} \tilde{L} d\sigma = 4\pi L_0 + \cancel{\int \omega \cdot L_1}(\text{uniform w/respect to } \omega) = 4\pi L_0$$

$$L_0 = \frac{1}{4\pi}\phi(x)$$

First moment becomes:

$$\text{Vector irradiance } \vec{E}(x) = \int_{4\pi} L(\omega)\omega d\sigma = \int_{4\pi} \tilde{L}(\omega)\omega d\sigma = \int_{4\pi} (L_0 + \omega \cdot L_1)\omega d\omega =$$

$$\cancel{L_0 \int \omega d\omega} \text{ (anti-symmetric)} + \int \omega \cdot (\omega \cdot L_1) d\omega =$$

$$\int (\omega\omega^\top) L_1 d\omega = \int (\omega\omega^\top) d\omega L_1 = \frac{4\pi}{3} L_1$$

$$L_1 = \frac{3}{4\pi}\vec{E}(x)$$

Combing these two results we get

$$\tilde{L}(x,\omega) = \frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega \cdot \vec{E}(x)$$

4.2 Zeroth-Order Equation

We substitute our new approximated value $\tilde{L}(x)$ in place of $L(x)$ into the equation for volumetric light transport:

$$(\omega \cdot \nabla)\tilde{L}(x,\omega) + \sigma_t \tilde{L}(x,\omega) = \sigma_s \int_{4\pi} p(\omega,\omega')\tilde{L}(x,\omega')d\omega' + Q(x,w)$$

$$(\omega \cdot \nabla)(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega \cdot \vec{E}(x)) + \sigma_t(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega \cdot \vec{E}(x)) = \sigma_s \int_{4\pi} p(\omega,\omega')(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega' \cdot \vec{E}(x))d\omega' + Q(x,w)$$

And then take the 0th moments of the left- and right-hand sides of the equation, yielding the 0th order approximation:

$$\begin{aligned}
\text{LHS} &= (\omega \cdot \nabla)(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega \cdot \vec{E}(x)) + \sigma_t(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega \cdot \vec{E}(x)) \\
&= \frac{1}{4\pi}\omega \cdot \nabla\phi(x) + \frac{3}{4\pi}(\omega \cdot \nabla)(\omega \cdot \vec{E}(x)) + \sigma_t(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega \cdot \vec{E}(x)) \\
&= \frac{1}{4\pi}\omega \cdot \nabla\phi(x) + \frac{3}{4\pi}\omega^\top(\nabla\vec{E}(x))\omega + \sigma_t(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega \cdot \vec{E}(x))
\end{aligned}$$

$$\begin{aligned}
\mu_0(\text{LHS}) &= \mu_0\left(\frac{1}{4\pi}\omega \cdot \nabla\phi(x) + \frac{3}{4\pi}\omega^\top(\nabla\vec{E}(x))\omega + \sigma_t(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega \cdot \vec{E}(x))\right) \\
&= \mu_0\left(\frac{1}{4\pi}\omega \cdot \nabla\phi(x)\right) + \mu_0\left(\frac{3}{4\pi}\omega^\top(\nabla\vec{E}(x))\omega\right) + \mu_0\left(\frac{\sigma_t}{4\pi}\phi(x)\right) + \mu_0\left(\frac{3\sigma_t}{4\pi}\omega \cdot \vec{E}(x)\right) \\
&= \frac{1}{4\pi}\underbrace{\mu_0\left(\omega \cdot \nabla\phi(x)\right)}_{\mu_0(\vec{a}\cdot\omega)=0 \text{ by } (5)} + \frac{3}{4\pi}\underbrace{\mu_0\left(\omega^\top(\nabla\vec{E}(x))\omega\right)}_{\mu_0(\omega^\top A\omega)=\frac{4\pi}{3}Tr(A) \text{ by } (7)} + \frac{1}{4\pi}\sigma_t\underbrace{\mu_0\left(\phi(x)\right)}_{\mu_0(C)=4\pi C \text{ by } (2)} + \frac{3\sigma_t}{4\pi}\underbrace{\mu_0\left(\vec{E}(x)\cdot\omega\right)}_{\mu_0(\vec{a}\cdot\omega)=0 \text{ by } (5)} \\
&= 0 + Tr(\nabla E) + \sigma_t\phi(x) + 0
\end{aligned}$$

$$\mu_0(\text{LHS}) = \nabla \cdot E + \sigma_t\phi(x) \tag{12}$$

$$\mu_0(\text{RHS}) = \mu_0\left(\sigma_s \int_{4\pi} p(\omega,\omega')(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\omega'\cdot\vec{E}(x))d\omega' + Q(x,w)\right)$$

$$= \frac{\sigma_s}{4\pi}\mu_0\left(\underbrace{\int_{4\pi} p(\omega,\omega')\overbrace{\phi(x)}^{\text{const.}}d\omega'}_{=\phi(x)\text{ (because }\int_{4\pi}p(\omega,\omega')d\omega'=1)}\right) + \frac{3\sigma_s}{4\pi}\mu_0\left(\underbrace{\int_{4\pi} p(\omega,\omega')\omega' d\omega'}_{=\int_{4\pi}p(\omega,\omega')\omega'd\omega'=\mu_1(p(\omega,\omega'))=g\omega\text{ by (10)}} \cdot \overbrace{\vec{E}(x)}^{\text{const.}}\right) + \mu_0\left(Q(x,\omega)\right)$$

$$= \frac{\sigma_s}{4\pi}\underbrace{\mu_0\left(\phi(x)\right)}_{\mu_0(C)=4\pi C\text{ by (2)}} + \frac{3\sigma_s}{4\pi}\underbrace{\mu_0\left(g\omega\cdot\vec{E}(x)\right)}_{\mu_0(\vec{a}\cdot\omega)=0\text{ by (5)}} + Q_0(x,\omega)$$

$$\mu_0(\text{RHS}) = \sigma_s\phi(x) + Q_0(x,\omega) \tag{13}$$

Equating the moments of the two sides, we get the zeroth order approximation:

$$\nabla\cdot\vec{E}(x) + \sigma_t\phi(x) = \sigma_s\phi(x) + Q_0(x) \tag{14}$$

Or equivalently,

$$\nabla\cdot\vec{E}(x) = (\sigma_s - \sigma_t)(x)\phi(x) + Q_0(x,\omega) = -\sigma_a(x)\phi(x) + Q_0(x)$$

The resulting statement tells us that the vector irradiance expresses the net flow of power across a surface and the divergence of that says how much it's flowing into or out of an area. Power flows out of areas where there are sources, and into (disappears from) areas that have absorption.

4.3 First-Order Equation

We will examine the moments of the two sides of the equation separately. First, the left hand side:

$$
\begin{aligned}
LHS &= \mu_1\left[(\omega\cdot\nabla)L(\vec{x},\omega)+\sigma_t L(\vec{x},\omega)\right]\\
&\left(\text{substituting the approximation } \frac{1}{4\pi}\phi(\vec{x})+\frac{3}{4\pi}\omega\cdot\vec{E}(\vec{x}) \text{ for } L(\vec{x},\omega)\text{:}\right)\\
&= \mu_1\left[(\omega\cdot\nabla)\left(\frac{1}{4\pi}\phi(\vec{x})+\frac{3}{4\pi}\omega\cdot\vec{E}(\vec{x})\right)\right]+\mu_1\left[\sigma_t\left(\frac{1}{4\pi}\phi(\vec{x})+\frac{3}{4\pi}\omega\cdot\vec{E}(\vec{x})\right)\right]\\
&\text{(by applying (11) and (12) to the first two terms:)}\\
&= \frac{1}{4\pi}\underbrace{\mu_1(\omega\cdot\nabla\phi(\vec{x}))}_{\mu_1(\omega\cdot a)=\frac{4\pi}{3}a\ \text{by}(6)}+\frac{3}{4\pi}\underbrace{\mu_1(\omega^T\nabla\vec{E}(\vec{x})\omega)}_{\mu_1(\omega^T A\omega)=0\ \text{by}(8)}+\frac{1}{4\pi}\underbrace{\mu_1(\phi(\vec{x})\sigma_t)}_{\mu_1(c)=0\ \text{by}(3)}+\frac{3}{4\pi}\underbrace{\mu_1(\omega\cdot\vec{E}(\vec{x})\sigma_t)}_{\mu_1(\omega\cdot a)=\frac{4\pi}{3}a\ \text{by}(6)}\\
&\text{(applying the specified lemmas about moments to each term:)}\\
&= \frac{1}{4\pi}\frac{4\pi}{3}\nabla\phi(\vec{x})+0+0+\frac{3}{4\pi}\frac{4\pi}{3}\sigma_t\vec{E}(\vec{x})\\
&= \frac{1}{3}\nabla\phi(\vec{x})+\sigma_t\vec{E}(\vec{x})
\end{aligned}
$$

For the right hand side:

$$
RHS = \frac{\sigma_s}{4\pi}\mu_1\left[\int\rho(\vec{x},\omega,\omega')\phi(\vec{x})d\omega'\right]+\frac{3\sigma_s}{4\pi}\mu_1\left[\int\rho(\vec{x},\omega,\omega')\omega'\cdot\vec{E}(\vec{x})d\omega'\right]+\underbrace{Q_1(\vec{x})}_{Q_1(\vec{x})=\mu_1(Q(\vec{x}))}
$$

Note that both $\phi(\vec{x})$ and $\vec{E}(\vec{x})$ are constants in their respective integrals. Therefore in the first case we are integrating the value of the distribution over the entire sphere which, by definition, has to be one. We are left with the first moment of the constant function 1, which by (3) is 0. For the second term, we can move the $\vec{E}(\vec{x})$ outside of the integral, leaving us with:

$$
\int\rho(\vec{x},\omega,\omega')\omega' d\omega'\cdot\vec{E}(\vec{x})
$$

and, by definition, the integral is the first moment of the phase function, which by (10) is $g\omega$. Substituting back into the equation for the right hand side:

$$
\begin{aligned}
RHS &= \vec{0}+\frac{3\sigma_s}{4\pi}\underbrace{\mu_1\left[g\omega\cdot\vec{E}(\vec{x})\right]}_{\mu_1(\omega\cdot a)=\frac{4\pi}{3}a\ \text{by}(6)}+Q_1(\vec{x})\\
&= \frac{3\sigma_s}{4\pi}\frac{4\pi}{3}g\vec{E}(\vec{x})+Q_1(\vec{x})\\
&= \sigma_s g\vec{E}(\vec{x})+Q_1(\vec{x})
\end{aligned}
$$

Now, equating the two sides gives us:

$$\begin{aligned}\frac{1}{3}\nabla\phi(\vec{x}) + \sigma_t\vec{E}(\vec{x}) &= \sigma_s g\vec{E}(\vec{x}) + Q_1(\vec{x}) \\ \frac{1}{3}\nabla\phi(\vec{x}) &= (-\sigma_a - \sigma_s + g\sigma_s)\vec{E}(\vec{x}) + Q_1(\vec{x}) \\ \frac{1}{3}\nabla\phi(\vec{x}) &= -(\sigma_a + \underbrace{(1-g)\sigma_s}_{\sigma_s'})\vec{E}(\vec{x}) + Q_1(\vec{x})\end{aligned}$$

σ_s' is called the reduced scattering coefficient. Note that this is the only place σ_S or g appear in the equation, meaning that altering σ_s and g have the same effect. Defining $\sigma_t' = \sigma_a + \sigma_s'$ to be the reduced extinction coefficient, we can re-write the first-order equation as:

$$\frac{1}{3}\nabla\phi(x) = -\sigma_t'\vec{E}(\vec{x}) + Q_1(\vec{x}) \tag{15}$$

4.4 The End: A Differential Equation

We can now combine the zero- and first-order equations to get a differential equation that we can actually use:

If sources are isotropic $Q_1(\vec{x}) = 0$, and (15) simplifies to: $\nabla\phi(\vec{x}) = -3\sigma_t'(\vec{x})\vec{E}(\vec{x})$ and solving for $\vec{E}(\vec{x})$ we get: $\vec{E}(\vec{x}) = -\frac{1}{3\sigma_t'(\vec{x})}\nabla\phi(\vec{x})$ or just $\vec{E} = -\frac{1}{3\sigma_t'}\nabla\phi$.

Substituting this into the zero-order equation (14), $(\nabla \cdot \vec{E}) = -\sigma_a\phi + Q_0$ yields:

$$\begin{aligned}\nabla \cdot (-\frac{1}{3\sigma_t'}\nabla\phi) &= -\sigma_a\phi + Q_0 \\ -\frac{1}{3\sigma_t'}\nabla^2\phi &= -\sigma_a\phi + Q_0 \\ \nabla^2\phi &= 3\sigma_a\sigma_t'\phi - 3\sigma_t'Q_0\end{aligned}$$

Alternatively, if we were to keep the Q_1 term ($\vec{E} = -\frac{1}{3\sigma_t'}\nabla\phi + \frac{1}{\sigma_t'}Q_1$) the substitution will yield:

$$\begin{aligned}-\frac{1}{3\sigma_t'}\nabla^2\phi + \frac{1}{\sigma_t'}Q_1 &= -\sigma_a\phi + Q_0 \\ D\nabla^2\phi &= \sigma_a\phi - Q_0 + 3D\nabla \cdot Q_1\end{aligned}$$

where $D = \frac{1}{3\sigma_t'}$. Note that this is the form used in the [Jensen et al.] paper.