CS667 **Lecture 12: Volume Rendering Equation** 3 March 2004

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1 The Scattering Equation

Last class, we derived the scattering equation:

$$(\omega \cdot \nabla)L(x,\omega) + \sigma_t(x)L(x,\omega) = \epsilon(x,w) + \sigma_s(x)\int_{S^2} p(x,\omega,\omega')L(x,\omega')\, d\omega' \tag{1}$$

This equation is an integro-differential and is not easy to evaluate. However, if we restrict the solution of the equation, as in Figure 1, to find only the radiance at a point, x, in a particular direction, ω_0. Then the equation only depends on the points along the ray that connects x to some point $y = \Psi(x, -\omega_0)$ with emittance, L_e, and where $\Psi(x,\omega)$ is the ray casting function[1]. In this domain, the equation can be solved as an ordinary differential equation.

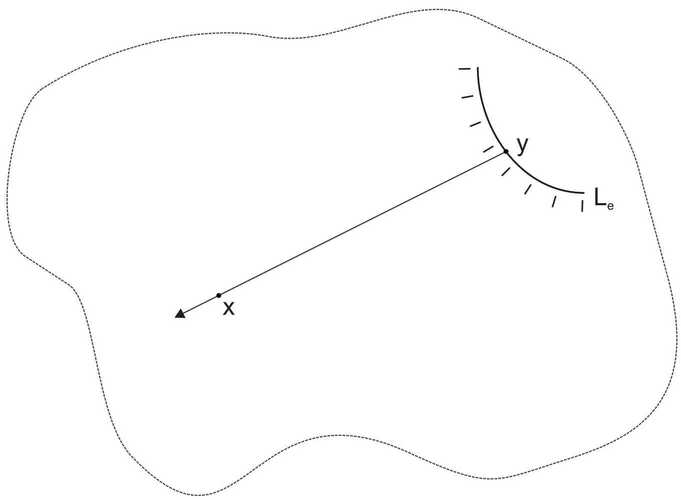

Figure 1: Restrict the solution of the scattering equation to a simpler domain, the ray between x and y.

2 Derivation of the Volume Rendering Equation

First, we must parameterize x as the distance t from a point y in direction ω_0. Since the direction of the ray is fixed, ω_0, the important functions in the scattering equation are reduced to smaller functions of t.

$$\begin{aligned}
x(t) &= y + \omega_0 t \\
L(t) &= L(x(t), \omega_0) \\
L(t,\omega') &= L(x(t), \omega') \\
\sigma_t(t) &= \sigma_t(x(t)) \\
\epsilon(t) &= \epsilon(x(t), \omega_0) \\
\sigma_s(t) &= \sigma_s(x(t)) \\
p(t,\omega') &= p(x(t), \omega_0, \omega')
\end{aligned}$$

[1] $\Psi(x,\omega)$ is a function that returns the first visible object from x in direction ω.

The scattering equation can then be written as:

$$L'(t) + \sigma_t(t)L(t) = \epsilon(t) + \sigma_s(t) \int_{S^2} p(t,\omega')L(t,\omega')\ d\omega' \tag{2}$$

Even though the right hand side of (2) still contains an integral, it's a definite integral over the sphere and only has a dependence on t and not ω' and more importantly, it does not depend on $L(t)$, i.e. $L(t, \omega_0)$ only makes and infinitesimal (measure 0) contribution to the integral. For simplicity, we can replace the entire right hand side of (2) with a single function of t.

$$q(t) = \epsilon(t) + \sigma_s(t) \int_{S^2} p(t,\omega')L(t,\omega')\ d\omega'$$

Finally, letting $y(t) = L(t)$ and $p(t) = \sigma_t(t)$ and dropping the function parameters, (2) becomes simply:

$$y' + py = q \tag{3}$$

2.1 Only Absorbtion: $q \equiv 0$

If the medium neither emits nor scatters light, $q \equiv 0$, and (3) reduces to the homogeneous equation:

$$y' + py = 0$$

The solution in the homogeneous case is relatively simple:

$$\begin{aligned}
y' + py &= 0 \\
\frac{y'}{y} &= -p \\
\int \frac{y'}{y}\ dt &= -\int p\ dt \\
ln\ y + c &= -\int p\ dt \\
y &= Ce^{-\int p\ dt}
\end{aligned}$$

Replacing the many variable substitutions that have been done, this final equation[2] becomes:

$$L(x,\omega) = Ce^{\oint_y^x \sigma_t(t)\ dt}$$

From the boundary conditions, $L(y,w) = L_e(y,w)$ and this defines the constant C.

$$L(y,\omega) = Ce^{\oint_y^y \sigma_t(t)\ dt} = Ce^0 = C = L_e(y,w)$$

The final homogeneous solution becomes:

$$L(x,\omega) = L_e(y,w)e^{\oint_y^x \sigma_t(t)\ dt} \tag{4}$$

The interpretation of this equation is that the radiance is the same as would be seen in the absence of the medium, $L_e(y,\omega)$, but attenuated by the absorbtion that occurs between the surface point, y, and the observation point, x.

[2] The term $\oint_y^x \sigma_t(t)\ dt$ in this equation is shorthand for the integral of $\sigma_t(t)$ on the straight line path from the point y to the point x, or more precisely, $\int_0^{\|x-y\|} \sigma_t(y + \omega t)\ dt$

2.2 Absorbtion and Scattering

To solve the complete version of (3), we will find functions μ and g so that the original equation can be rewritten as

$$(\mu y)' = g$$

By expanding the above:

$$\begin{aligned} (\mu y)' &= g \\ \mu y' + \mu' y &= g \\ y' + \frac{\mu'}{\mu} y &= \frac{g}{\mu} \end{aligned}$$

And equating these coefficients with those of (3) and some calculus:

$$\begin{aligned} p = \frac{\mu'}{\mu} &\quad \Rightarrow \quad \mu = e^{\int p\ dt} \\ q = \frac{g}{\mu} &\quad \Rightarrow \quad g = \mu q = q e^{\int p\ dt} \end{aligned}$$

With these coefficients in hand, we can solve:

$$\begin{aligned} (\mu y)' &= g \\ \mu y &= \int g\ dt + C \\ y &= \frac{\int g\ dt + C}{\mu} \\ y &= \frac{\int g\ dt + C}{\mu} \end{aligned}$$

With the substitution of the original values and a change of variable names to preserve the correct evaluation of the nested integrals, this becomes:

$$\begin{aligned} y &= \frac{\int g\ dt + C}{\mu} \\ y(t) &= \frac{\int_0^t q(x) e^{-\int_0^x p(x')\ dx'}\ dx + C}{e^{\int_0^t p(x')\ dx'}} \\ y(t) &= \int_0^t q(x) e^{-\int_0^x p(x')\ dx' + \int_0^t p(x')\ dx'}\ dx + C e^{-\int_0^t p(x')\ dx'} \\ y(t) &= \int_0^t q(x) e^{-\int_t^x p(x')\ dx'}\ dx + C e^{-\int_0^t p(x')\ dx'} \end{aligned}$$

Finally, the original substitutions can be replaced and the boundary conditions can be used to set C (As in the homogeneous case, C equals the light emitted from y in ω).

$$L(x,\omega) = \int_y^x e^{-\int_x^{x'} \sigma_t(x'')\, dx''} \left(\epsilon(x') + \sigma_s(x') \int_{S^2} p(x',\omega',\omega) L(x',\omega')\, d\omega' \right)\, dx' + L_e(y,\omega) e^{-\int_x^y \sigma_t(x')\, dx'} \quad (5)$$

However this form is not very intuitive. Lets define a attenuation function that describes the loss of light due to both absorbtion and scattering from a point y to a point x:

$$\alpha(x,y) = e^{-\int_x^y \sigma_t(x,\,)dx}$$

With this function and some rearragement of (5), we get the final form of the Volume Rendering Equation:

$$L(x,\omega) = \int_y^x \alpha(x,x')\epsilon(x')\, dx' + \int_x^y \alpha(x,x')\sigma_s(x') \int_{S^2} p(x',\omega',\omega) L(x',w')\, d\omega'\, dx' + \alpha(x,y) L_e(y,\omega) \quad (6)$$

The three terms of (6) each represent a different contribution to the final rediance. The first term is the radiance emitted from the field that reaches x in direction ω. The second term represents the radiance scattered from the field towards x. Finally, the third term represents the emitted radiance from the surface at y that reaches x. All terms are attentuated dependent on their distance from x. A much closer analog to the orignal rendering equation can be seen with a few more renamings. Let:

$$\begin{aligned} E(x,\omega) &= \int_y^x \alpha(x,x')\epsilon(x')\, dx' + \alpha(x,y) L_e(y,\omega) \\ K(x,x',\omega,\omega') &= \alpha(x,x')\sigma_s(x') p(x',\omega',\omega) \end{aligned}$$

Then rewrite (6) as:

$$L(x,\omega) = E(x,\omega) + \int_y^x \int_{S^2} K(x,x',\omega,\omega') L(x',\omega')\, d\omega'\, dx'; \quad (7)$$

In this final form, the analogy to the original rendering equation becomes clear. The radiance at a point x in direction ω equals the sum of E, the radiance emitted from x in ω, plus the integral of radiance over all other points and directions that contribute to the radiance at x in ω scaled by some transfer function K.