CS667 **Lecture 4: Point Based Rendering Equation** 5 February 2004

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1 Changing the Domain of Integration

In this lecture, we formulate the rendering equation from points to points instead of our usual hemisphere formulation.

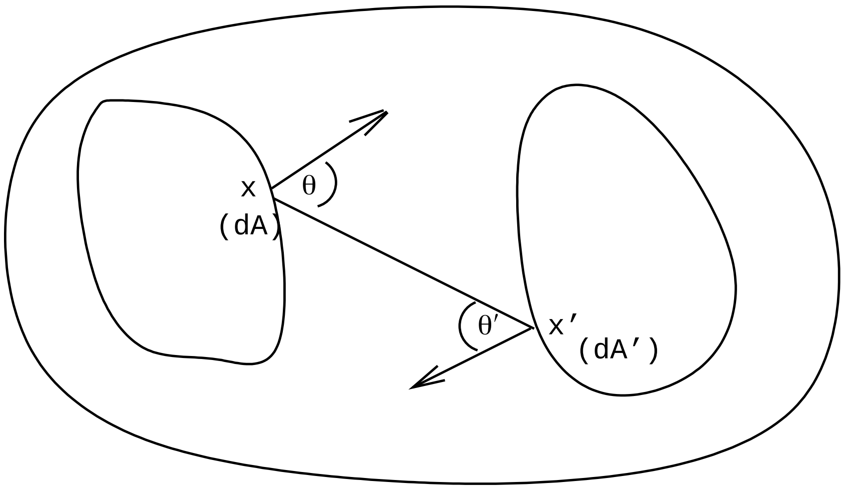

Figure 1: Two points on two surfaces.

That is, instead of the rendering equation yielding flux per unit area, per unit solid angle, we now will formulate the equation to yield flux per unit area, per unit area. Formally:

$$L(x' \to x) = \frac{d^2\Phi}{dA^{\perp} d\omega} = \frac{d^2\Phi}{dA \cos\theta \frac{dA'}{r^2} \cos\theta'}$$

is reformulated as I:

$$I(x' \to x) = \frac{d^2\Phi}{dA dA'}$$

so it's clear that

$$I = L\frac{\cos\theta\cos\theta'}{r^2}$$

So now, to compute irradiance, for example, we can integrate over the domain of all surfaces $\mathcal{M}$:

$$E(x) = \int_{\mathcal{M}} I(x' \to x) dA(x')$$

So, $I : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ (with units $\frac{W}{m^4}$)

2 Transport Reflectance ρ

In order to create a point formulation, instead of using the BRDF we now consider the *3-point transport reflectance* ρ.

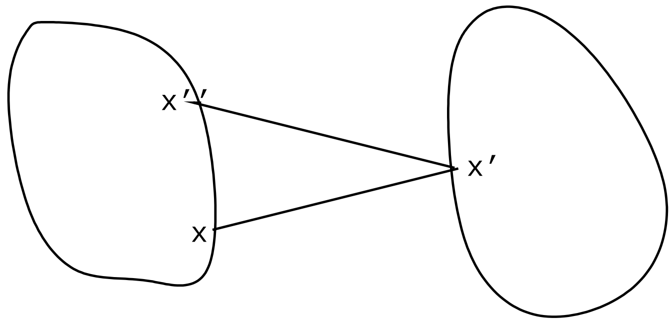

Figure 2: Three points on two surfaces.

Now we can define $I(x' \to x)$ in terms of I and the 3-point transport reflectance from x'' to x' to x, $\rho(x'' \to x' \to x)$:

$$I(x' \to x) = \frac{1}{r^2} \int_{\mathcal{M}} \rho(x'' \to x' \to x) I(x'' \to x') dA(x'')$$

where $r = \|x' - x\|$. So, we are integrating over all surfaces the transport between the three points times the incoming radiosity to our current point x' over all incoming points x''. (Assuming x' and x are visible from one another.) Surprisingly, ρ is unitless. This is because we multiply by area as we integrate over illuminating surfaces, and then divide by area due to the $\frac{1}{r^2}$ falloff. Hence, the units of I are:

$$m^{-2} \int 1 \cdot \frac{W}{m^4} \cdot m^2 = \frac{W}{m^4}$$

3 ρ for Lambertian Surfaces

We will now derive ρ for a Lambertian reflector. Consider the following setup:

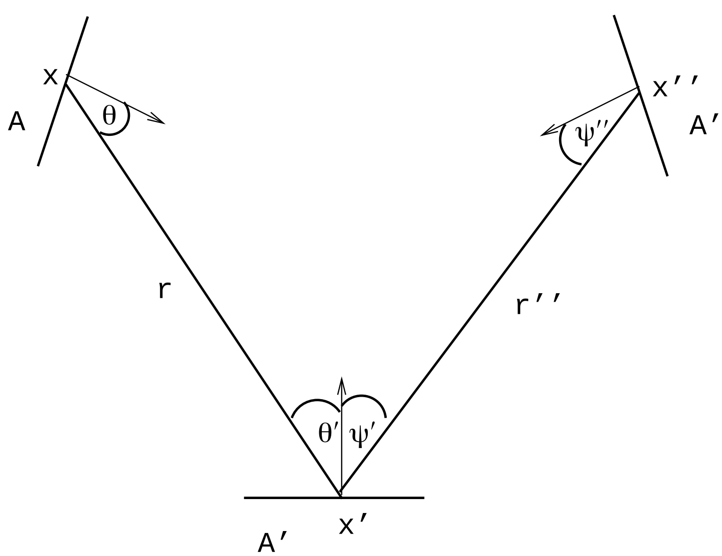

Figure 3: Three surfaces with points x, x', and x''

Consider the equations for the Lambertian surface A' of irradiance, radiosity, and radiance, respectively.

$$E(x') = \frac{d\Phi_{\to x'}}{dA'} = \int_{\mathcal{M}} I(x'' \to x') dA(x'')$$

$$M(x') = \frac{d\Phi_{x' \to}}{dA'} = R_{A'} E(x')$$

$$L(x') = \frac{d^2\Phi_{x'\to}}{dA'd\omega} = \frac{R_{A'}}{\pi}E(x')$$

Now, to find $I(x' \to x)$, note that the projected solid angle subtended by A is $\mu(A) = \frac{A\cos\theta\cos\theta'}{r^2}$. So the flux incident on A per unit area of A' is

$$\frac{d\Phi_A}{dA'} = A\frac{\cos\theta\cos\theta'}{r^2}\frac{R_{A'}}{\pi}\int_{\mathcal{M}} I(x'' \to x')dA(x'')$$

Now, differentiate with resepect to A

$$I(x' \to x) = \frac{d^2\Phi_A}{dAdA'} = \frac{1}{r^2}\int_{\mathcal{M}} \frac{R_{A'}\cos\theta\cos\theta'}{\pi} I(x'' \to x')dA(x'')$$

So, it's clear that the $\frac{R_{A'}\cos\theta\cos\theta'}{\pi}$ term is $\rho(x'' \to x' \to x)$ (for Lambertian surfaces.) It is dependent on x for a particular x', and is independent of x''. In terms of the BRDF:

$$\rho(x'' \to x' \to x) = f_r(\ldots)\cos\theta\cos\theta'.$$

4 Visibility and Emission

We're almost to the rendering equation, but we haven't accounted for visibility and light emission. For occlusion, we must replace the $\frac{1}{r^2}$ term with a *geometry factor* that accounts for when surfaces cannot see one another:

$$g(x, x') = \begin{cases} \frac{1}{\|x-x'\|} & \text{if } x, x' \text{ are visible to one another;} \\ 0 & \text{otherwise.} \end{cases}$$

Light emission is introduced by a *transport emittance* function:

$$\epsilon(x' \to x) = L_e^{\circ}(x', \vec{xx'})\cos\theta\cos\theta'$$

So the units for ϵ are $\frac{W}{m^2}$. Due to falloff, to compute the transport intensity due to a particular transport emittance, we must divide by r^2

$$I(x' \to x) = \frac{1}{r^2}\epsilon(x' \to x)$$

In order to compensate for occlusion from emitters as well, we can put the geometric term around the rest of the rendering equation, instead of inside the integral over surfaces

$$I(x' \to x) = g(x, x')\left[\int_{\mathcal{M}} \rho(x'' \to x' \to x)I(x'' \to x')dA(x'') + \epsilon(x' \to x)\right]$$

As a check, the units of I stay the same:

$$m^{-2}\left[\int_{\mathcal{M}}\left(1 \cdot \frac{W}{m^4} \cdot m^2\right) + \frac{W}{m^2}\right] = \frac{W}{m^4}$$

5 Point Based and Hemisphere Based Symmetry

We can now begin to more clearly see the relationship between the hemispherical and point formulations of the rendering equation. If

$$V(x, x') = \begin{cases} 1 & \text{if } x, x' \text{ are visible to one another;} \\ 0 & \text{otherwise.} \end{cases}$$

In terms of radiance,

$$
\begin{aligned}
I(x' \to x) &= L(x' \to x)\frac{\cos\theta\cos\theta'}{r^2} \\
&= \frac{V(x,x')}{r^2}\left[L_e^\circ(x' \to x)\cos\theta\cos\theta' + \int_{\mathcal{M}} \rho(x'' \to x' \to x)\cos\theta\cos\theta' L(x'' \to x')\frac{\cos\psi''\cos\psi'}{r''^2}dA(x'')\right]
\end{aligned}
$$

Now, if we cancel the $\frac{\cos\theta\cos\theta'}{r^2}$ term,

$$
L(x' \to x) = V(x,x')\left[L_e^\circ(x' \to x) + \int_{\mathcal{M}} \rho(x'' \to x' \to x)L(x'' \to x')G(x',x'')dA(x'')\right]
$$

When compared to the hemispherical formulation

$$
L_e(x',\omega) = L_e^\circ(x',\omega) + \int_{\mathcal{H}^2} f_r(x',\omega,x'')L_i(x,\omega'')d\mu(\omega'')
$$

The symmetry becomes clear: there is a difference in the domain and measure of the integral. In fact, we notice that

1. $V(x,x')\int_{\mathcal{M}} \cdots dA(x')$ is an integral over the hemisphere of all visible surfaces
2. $G(x,x')dA(x')$ is the projected solid angle measure. That is, $d\mu(\omega'')$ is equivalent to $\frac{\cos\psi''\cos\psi'}{r''^2}dA(x'')$

Hence, the difference between the solid angle and area formulations is merely a change of variable!