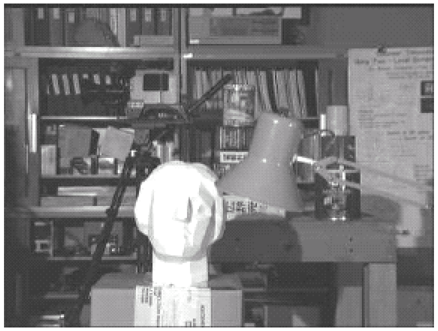
CS 664
Belief Propagation for Early Vision

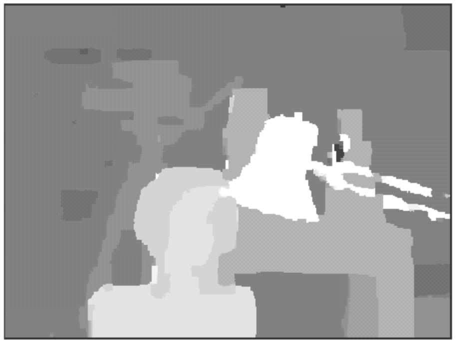
Daniel Huttenlocher

Cornell University
Faculty of Computing and Information Science

Low Level Vision Problems

- Estimate label at each pixel
  - Stereo: disparity
  - Restoration: intensity
  - Segmentation: layers, regions
  - Optical flow: motion vector

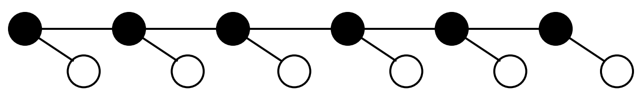


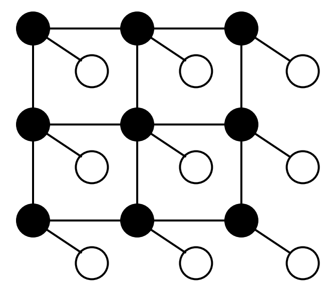
Pixel Labeling Problem

- Find good assignment of labels to sites
  - Set $\mathcal{L}$ of k labels
  - Set $\mathcal{S}$ of n sites
  - Neighborhood system $\mathcal{N} \subseteq \mathcal{S} \times \mathcal{S}$ between sites
    - Consider case of (four connected) grid graph
- Undirected graphical model
  - Graph $\mathcal{G} = (\mathcal{S}, \mathcal{N})$
  - Discrete random variable x_i over $\mathcal{L}$ at each site i
  - First order models
    - Maximal cliques in $\mathcal{G}$ of size 2

Markov Random Field

- Labels are values of hidden states, x_i
  - Not observable
  - Posterior probability of labels given observed data, o_i
- Reachability in graph corresponds to conditional dependence of random variables
- 1D: hidden Markov model





Form of Posterior

- Observations o, hidden states x
- Posterior distribution of labelings given observations

$$\Pr(x|o) \propto \Pr(o|x)\Pr(x)$$

- For first order model, prior factors as

$$\Pr(x) \propto \prod_{(i,j)\in\mathcal{N}} V(x_i,x_j)$$

- Further assume likelihood factors

$$\Pr(x|o) \propto \prod_{i\in\mathcal{S}} D_i(x_i) \prod_{(i,j)\in\mathcal{N}} V(x_i,x_j)$$

Estimation Problems

- Marginal probability at each node

$$\Pr(x_i | o)$$

- Maximize posterior (MAP)

$$\text{argmax}_x \prod_{i \in \mathcal{S}} D_i(x_i) \prod_{(i,j) \in \mathcal{N}} V(x_i, x_j)$$

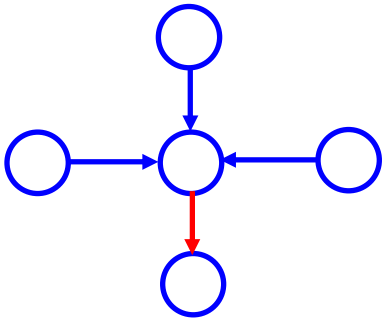
- Not computationally tractable
  - NP hard for 3 or more labels and robust V
- Various methods for approximate solution
  - Annealing, variational techniques, graph cuts using α-expansion, loopy belief propagation, …

Belief Propagation

- Iterative local update technique
  - Message passing, “nosy neighbor”
- Two forms
  - Sum product for estimating marginals
  - Max product for MAP estimation
- Exact solution when no loops in graph
- Update messages until “convergence” then compute distribution at each node
  - Sum product for marginals
  - Max product then max at each node for MAP

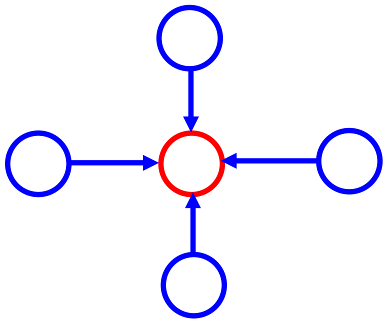
Sum Product

- At each step node j sends each neighbor a message, in parallel
  - Node j's view of i's labels



$$m_{j\to i}(x_i) = \sum_{x_j}(D_j(x_j)\, V(x_j,x_i) \prod_{k\in\mathcal{N}(j)\setminus i} m_{k\to j}(x_j))$$

- After T iterations compute belief at each node
  - Using messages from neighbors and local data



$$b_j(x_j) = D_j(x_j) \prod_{i\in\mathcal{N}(j)} m_{i\to j}(x_j)$$

Max Product

- Min sum form with cost functions D',V' proportional to negative log potentials
- Message updates

$$m'_{j\to i}(x_i) = \min_{x_j}(D'_j(x_j) + V'(x_j,x_i) + \sum_{k\in\mathcal{N}(j)\backslash i} m'_{k\to j}(x_j))$$

- After T iterations compute label minimizing value at each node

$$\operatorname{argmin}_{x_j} (D'_j(x_j) + \sum_{i\in\mathcal{N}(j)} m'_{i\to j}(x_j))$$

  – Simple approach of separately minimizing at each node can be problematic

Three Techniques

- Memory requirements of BP large
  - Using bipartite form of graph can halve usage
- For vision problems $V(x_i, x_j)$ generally function of difference between labels
  - Enables computation of (discrete) messages in linear rather than quadratic time
- Number of iterations generally proportional to diameter of graph
  - Propagate information across grid
  - Using multi-grid methods can reduce to small constant number

Bipartite Graph ("Red-Black")

- Checkerboard pattern on grid defines a bipartite graph, $V=A \cup B$
- Alternating message updates of sets A,B yields messages $\overline{m}$ nearly same as m
  - Update messages from A on odd iterations and from B on even iterations
  - Then can show by induction when t odd (even)

$$\overline{m}^t_{i\to j} = \begin{cases} m^t_{i\to j} & \text{if i in A (i in B)} \\ m^{t-1}_{i\to j} & \text{otherwise} \end{cases}$$

  - Converges to same fixed point with half as many updates and half as much memory

Fast Message Updates

- Pairwise term V measuring label <u>difference</u>
- Sum product
  - Express as a convolution
  - O(klogk) algorithm using the FFT
  - Linear-time approximation algorithms for Gaussian models
- Min sum (max product)
  - Express as a min convolution
  - Linear time algorithms for common models using distance transforms and lower envelopes

Sum Product Message Passing

- When $V(x_i,x_j)=\rho(x_i-x_j)$ can write message update as convolution

$$m_{j\to i}(x_i) = \sum_{x_j}(\rho(x_j-x_i)\ h(x_j))$$
$$= \rho \star h$$

  - Where $h(x_j)= D_j(x_j) \prod_{k\in\mathcal{N}(j)\setminus i} m_{k\to j}(x_j))$
- Thus FFT can be used to compute in O(klogk) time for k values
  - Still somewhat large constants
- For ρ a (mixture of) Gaussian(s) do faster

Fast Gaussian Convolution

- A box filter has value 1 in some range

$$b_w(x) = \begin{cases} 1 \text{ if } 0 \le x \le w \\ 0 \text{ otherwise} \end{cases}$$

- A Gaussian can be approximated by repeated convolutions with a box filter
  - Application of central limit theorem, convolving pdf's tends to Gaussian
  - In practice, 4 convolutions [Wells, PAMI 86]

$$b_{w_1}(x) \ast b_{w_2}(x) \ast b_{w_3}(x) \ast b_{w_4}(x) \approx G_\sigma(x)$$

  - Choose widths w_i such that $\sum_i (w_i^2 - 1)/12 \approx \sigma^2$

Convolution Using Box Sum

- Thus can approximate $G_\sigma(x) \star h(x)$ by cascade of box filters

$$b_{w_1}(x) \star (b_{w_2}(x) \star (b_{w_3}(x) \star (b_{w_4}(x) \star h(x))))$$

- Compute each $b_w(x) \star f(x)$ in time independent of box width w – sliding sum
  - Each successive shift of $b_w(x)$ w.r.t. f(x) requires just one addition and one subtraction
- Overall computation just a few operations per label, O(k) with very low constant

Max Product Message Passing

- Can write message update as

$$m'_{j\to i}(x_i) = \min_{x_j}(\rho'(x_j - x_i) + h'(x_j))$$

  - Where $h'(x_j) = D'_j(x_j) \sum_{k\in\mathcal{N}(j)\setminus i} m'_{k\to j}(x_j))$
  - Formulation using minimization of costs, proportional to negative log probabilities
- Convolution-like operation over min,+ rather than $\sum,\times$ [FH00,FHK03]
  - No general fast algorithm like FFT
  - Certain important special cases in linear time

Commonly Used Pairwise Costs

- Potts model $\rho'(x) = \begin{cases} 0 \text{ if } x=0 \\ d \text{ otherwise} \end{cases}$
- Linear model $\rho'(x) = c|x|$
- Quadratic model $\rho'(x) = cx^2$
- Truncated models
  - Truncated linear $\rho'(x)=\min(d,c|x|)$
  - Truncated quadratic $\rho'(x)=\min(d,cx^2)$
- Min convolution can be computed in linear time for any of these cost functions

Potts Pairwise Model

- Substituting in to min convolution

  $$m'_{j \to i}(x_i) = \min_{x_j}(\rho'(x_j - x_i) + h'(x_j))$$

  can be written as

  $$m'_{j \to i}(x_i) = \min(h'(x_i), \min_{x_j} h'(x_j) + d)$$

- No need to compare pairs x_i, x_j
  - Compute min over x_j once, then compare result with each x_i
- O(k) time for k labels
  - No special algorithm, just rewrite expression to obtain alternative (fast) computation

Linear Pairwise Model

- Substituting in to min convolution yields

$$m'_{j \to i}(x_i) = \min_{x_j}(c|x_j - x_i| + h'(x_j))$$

- Similar form to the L_1 distance transform

$$\min_{x_j}(|x_j - x_i| + 1(x_j))$$

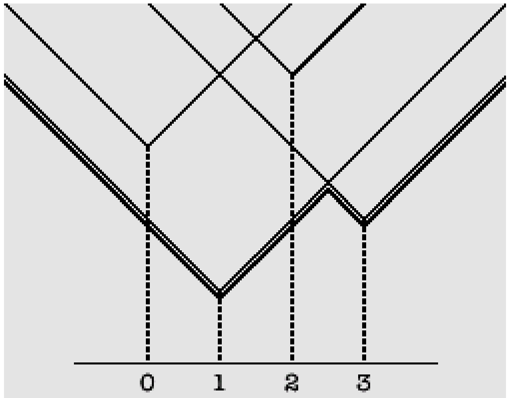
  – Where $1(x) = \begin{cases} 0 & \text{when } x \in P \\ \infty & \text{otherwise} \end{cases}$

  is an indicator function for membership in P

- Distance transform measures L_1 distance to nearest point of P
  – Can think of computation as lower envelope of cones, one for each element of P

Using the L_1 Distance Transform

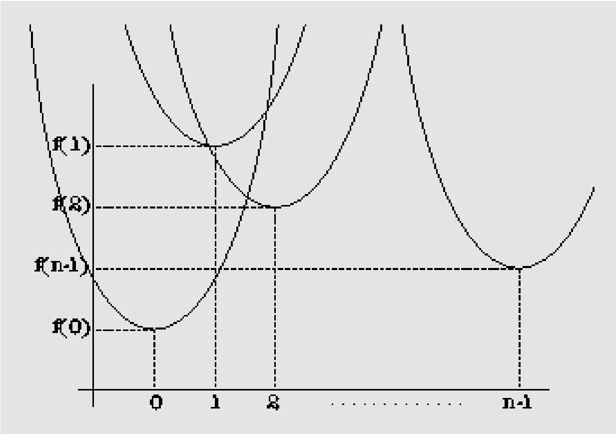
- Linear time algorithm
  - Traditionally used for indicator functions, but applies to any sampled function



- Forward pass
  - For x_j from 1 to k-1
    $m(x_j) \leftarrow \min(m(x_j), m(x_j\text{-}1)+c)$
- Backward pass
  - For x_j from k-2 to 0
    $m(x_j) \leftarrow \min(m(x_j), m(x_j+1)+c)$
- Example, c=1
  - (3,1,4,2) becomes (3,1,2,2) then (2,1,2,2)

Quadratic Pairwise Model

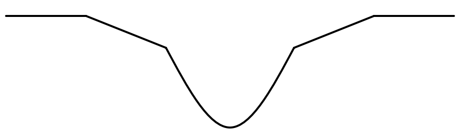
- Substituting in to min convolution yields
  $$m'_{j\to i}(x_i) = \min_{x_j}(c(x_j - x_i)^2 + h'(x_j))$$
- Again similar form to distance transform
- Compute lower envelope of parabolas
  - Each value of x_j defines a quadratic constraint, parabola rooted at $(x_j, h(x_j))$
  - In general can be done in O(klogk) [DG95]
  - Here parabolas are same shape and ordered, so O(k)



Combined Pairwise Models

- Truncated models
  - Compute un-truncated message m’
  - Truncate using Potts-like computation on m’ and original function h’

    $\min(m'(x_i), \min_{x_j} h'(x_j)+d)$
- More general combinations
  - Min of any constant number of linear and quadratic functions, with or without truncation
    - E.g., multiple “segments”



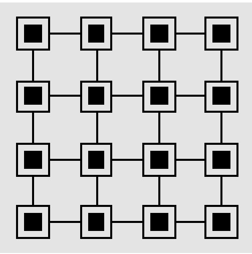
Fast Message Update Methods

- Efficient computation without assuming form of (discrete) distributions
  - Requires prior to be based on differences between labels rather than their identities
- Sum product
  - O(klogk) message updates for arbitrary discrete distributions over k labels using FFT
  - O(k) when pairwise clique potential a mixture of Gaussians using box sums
- Max product
  - O(k) for commonly used clique potentials

A Multi Grid Technique

- Number of message passing iterations T generally proportional to diameter of grid
  - Propagate information across the grid
- Use hierarchical approach to make independent of graph diameter
  - Previous work does this by changing the graph, building quad-tree with no loops [W02]
- Our approach is to define a hierarchy of problems with original graph structure
  - Initialize messages based on coarser levels

Hierarchy of Grids



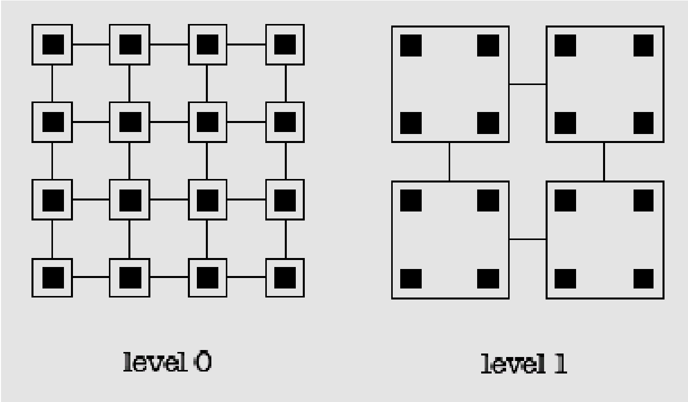
- Consider min sum case, rewrite minimization in terms of grid Γ

$$E(x) = \sum_{(i,j)\in\Gamma} D_{ij}(x_{i,j}) + \sum_{(i,j)\in\Gamma\backslash\mathcal{C}} V(x_{i,j}-x_{i+1,j}) + \sum_{(i,j)\in\Gamma\backslash\mathcal{R}} V(x_{i,j}-x_{i,j+1})$$

  – Where $\mathcal{C},\mathcal{R}$ last row and column of grid
- Can define family of grids Γ^0, Γ^1, ...
  – An element of Γ^ℓ corresponds to $\varepsilon\times\varepsilon$ block of pixels, where $\varepsilon=2^\ell$
  – Labeling x^ℓ of Γ^ℓ assigns the pixels in each block a single label (from same set $\mathcal{L}$)

Problem Hierarchy

- Minimization problem at each level of the hierarchy



$$E^{\ell}(x^{\ell}) = \sum_{(i,j)\in\Gamma^{\ell}} D^{\ell}_{ij}(x^{\ell}_{i,j}) + \sum_{(i,j)\in\Gamma^{\ell}\backslash\mathcal{C}^{\ell}} V^{\ell}(x^{\ell}_{i,j}-x^{\ell}_{i+1,j}) + \sum_{(i,j)\in\Gamma^{\ell}\backslash\mathcal{R}^{\ell}} V^{\ell}(x^{\ell}_{i,j}-x^{\ell}_{i,j+1})$$

- Multi grid: final messages at one level as initial condition for next level, and so on
  - Small number of iterations if initial conditions close to final value

Hierarchical Data Term

- Finite element approach
- Assigning label α to block (i,j) at level ℓ equivalent to assigning α to each pixel in block

$$D^{\ell}_{ij}(\alpha) = \sum_{0\leq u<\varepsilon}\sum_{0\leq v<\varepsilon} D_{\varepsilon i+u,\varepsilon j+v}(\alpha)$$

  – Sum costs for all pixels in block
- Corresponds to product of probabilities, likelihood of observing pixels given label α
- Captures preference for multiple labels

Hierarchical Discontinuity Term

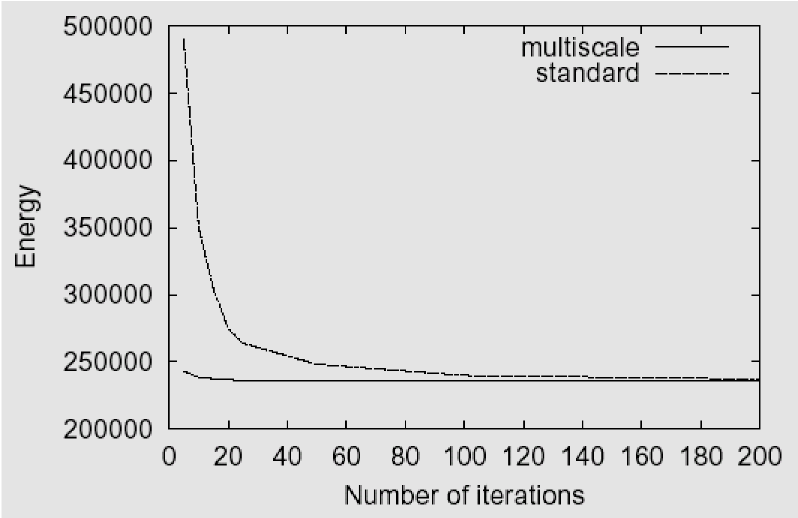
- Boundary between blocks length ε
  - Sum along boundary
- Separation between blocks ε
  - Finite difference, divide by separation

$$V^{\ell}(\alpha-\beta) = \varepsilon V\left(\frac{\alpha-\beta}{\varepsilon}\right)$$

- Produces different form depending on V
  - Linear, $V^{\ell}(x)=c|x|$
  - Quadratic, $V^{\ell}(x)=cx^2/\varepsilon$

Multi Grid Method

- Number of levels in hierarchy proportional to log image diameter
  – So propagation time small constant at top
- Same label set at each level
  – In contrast to pyramid methods
- In practice converges after a few iterations
  – Note each iteration just 1/3 more work than standard single level



Cornell University

Illustrative Results for Restoration

- Image restoration using MRF with truncated quadratic discontinuity cost
  - Not practical with conventional techniques, message updates 256^2
- Quadratic data term with no penalty for masked pixels
- Powerful formulation now practical
  - Largely abandoned except for small label sets

Gaussian noise and mask

Illustrative Results for Stereo

- Truncated linear cost functions

$$D_i(x_i) = \min(d_b, |L(p_{i1}, p_{i2}) - R(p_{i1} - x_i, p_{i2})|)$$

$$V(x_i, x_j) = \min(d_s, |x_i - x_j|)$$

  – Runs in under a second for 30 disparity levels
- Used for many of top methods in Middlebury stereo benchmark