Course notes, CS664, 11/9/04

- Quiz 3 put off by until Thursday 11/11, coverage will include OFCE (through previous lecture).

- By the end of today lecture each student should mail me your choice of a paper to write a 1-page report about (if it's available online, please include a URL). Two rules: must be related to the content of 664 and must not be a paper I covered or plan to cover. The second rule is slightly unfair, but hard to avoid...

- Next lecture (11/11) hand in a 1-paragraph project proposal. Projects are done 1 student at a time, and must have a research component (i.e., implementing or comparing existing methods isn't sufficient). Note: they are not required to work but must be implemented, and should do something reasonable at least some of the time. Also, they will be judged just on the writeup.

Intrinsic properties of curves

- Define a curve as $C(p) = (x(p), y(p)), p \in [0, 1]$. Tangent T is normalized (x_p, y_p), normal N is normalized $(y_p, -x_p)$. By convention N points inward as we move clockwise (other conventions are possible, and even textbooks sometimes are inconsistent about this).

- Typical active contour (snakes) energy function:

$$E(C) = \int_0^1 |C'(p)|^2 dp + \lambda \int_0^1 g(|\nabla I(C(p))|)^2 dp.$$

- g decreases, since you want high values of the gradient (i.e. edges) to have low energy.

- Typical form of g is

$$g(x, y) = \frac{1}{\sqrt{1 + |\nabla I(x, y)|^2}}$$

  Note this is 1 in flat regions and smaller closer to edges.

- The exact form of g isn't that important but this convention (in terms of where it's big and small) is actually quite useful, as we will see... In fact, g is often thought of as a potential function, sort of like height.

- Drawback: this is sensitive to the parameterization (i.e. speed we travel along the curve, plus starting/stopping point). If we replace g by ϕ where $\phi(r) = p, r \in [c, d]$, then the energy changes!

The first term becomes

$$\int_c^d |(C \circ \phi)'(r)|^2 (\phi'(r))^{-1} dr$$

and the second term changes similarly.

- Certain energy functions integrated along curves are insensitive to parameterization (Euler-Weierstrass)

- Consider the function $s(p)$ which is the length from 0 to p, defined by

$$s(p) = \int_0^q \sqrt{x_q^2(q) + y_q^2(q)} dq$$

Length of a curve is $L = \int_0^1 |C_p| dp$.

- Note that s can be used to reparameterize the curve, since it maps $[0, 1]$ to $[0, L]$. I.e., we can write $C(s) = C(p(s))$ by abuse of notation. You can verify that this parameterization has unit velocity (i.e. $|C_s| = 1$).

- This *arclength parameterization* has various nice properties. For instance you can prove that the curve and its derivatives (i.e., $C(s), C_s, C_{ss}$, etc.) are invariant to rotation and translation.

- The curvature can be written as a function of arclength $C_{ss} = \kappa(s)N$. Note that the radius of the osculating circle is $\frac{1}{\kappa}$.

- Properties of the curve that do not depend on parameterization or (sometimes) position or rotation are called *intrinsic*, or sometimes "geometric".

Curve evolution done wrong: snakes

- For curve evolution we will replace $C(p)$ by a family $C(p,t)$ where the input curve is $C(p) = C(p,0)$. You can think of a velocity vector field V that varies by space and time, acting on the curve.

- The slightly more general formulation of snakes is:

$$E(C) = \int_0^1 \alpha |C_p(p)|^2 + g(C(p))dp.$$

  (Can also have higher order derivatives, but this just adds complexity.)

- From Calculus of Variations, at a local extremum for E we will have

$$\alpha C_{pp}(p) + \nabla g(C(p)) = 0.$$

Curve evolution for snakes is

$$C_t = \alpha C_{pp} - \nabla g(C(p)),$$

which finds a local minimum of $E(C)$.

- This is ugly in many respects. Not the least of which is that it's completely non-geometric, so different parameterizations of the same curve will evolve differently!

Curve evolution done right

- The natural local coordinate system is defined by N, T. So the obvious curve evolution equation is of the form

$$C_t = V_N N + V_T T.$$

We can ignore the T term due to the Epstein-Gage theorem, which says that such a term only affects the parameterization. We will consider "geometric flows", i.e. V will be independent of parameterization.

- Obvious special case is constant V. Such curves are called *offsets* and related to dilation/erosion operators.

- Consider the thinning operation on a binary image (become 0 if any neighbor is 0). Doing this repeatedly on a shape will give you a curve-like representation that describes some of its essential topology (process is sometimes called skeletonization, or the medial axis).

- This can be done with curve evolution with offsets. We just force the curve to evolve inwards. Skeletonization will occur with "shocks" where the curve moves to intersect itself.

- In fact there is a closed form solution for the curvature over time given by (Ricatti equation)

$$\kappa(p,t) = \frac{\kappa(p,0)}{1 - t\kappa(p,0)}.$$

  This has singularities at $t = \frac{1}{\kappa}$ which is exactly when you expect to see the shock (since it is the radius of the osculating circle!)

Curvature flow

- Another important example is the "geometric heat equation" which tends to smooth out curves. It is $C_t = \kappa N$ (i.e. velocity vector field is the curvature).

- This takes any simple (non-self intersecting) curve first into a convex shape (Grayson) then into a circular point (Gage-Hamilton) which vanishes. Time to vanish is proportional to the area enclosed.

Geodesic active contours

- A really elegant solution that is close to the original snake energy function is to simply minimize

$$\int g(C(s))ds.$$

This looks too good to be true. Notice that the smoothness term is gone, replaced by the idea of minimizing the weighted length (shorter curves are smoother than longer ones). The weighting comes from g.

- The curve evolution equation that minimizes this is

$$C_t = (\kappa g - \langle \nabla g, N \rangle)N$$

- There are some nice physics connections here involving, e.g., light rays. Minimal length curves are very important, especially in physics (called *geodesics*).

- There is an equivalence between minimizing the weighted curve length and minimizing the original energy (oddly enough this equivalence depends on the data term and smoothness terms having identical weightings).

Shortest paths and minimal surfaces

- There are many papers that take a similar (but more complex) approach based on shorts paths (or minimial surfaces) where the actual metric varies over the space. This is very elegant but requires far more differential geometry than we can go into.

- A simple example is *intelligent scissors*, which does segmentation by shortest paths. Distances are small along links (in the graph) that point along edges (in the image). It is easy to turn the open-curve solution into a good closed-curve segmentation.

- There is even a cool relationship between this and graph cuts, which again is too complex for this course (Boykov/Kolmogorov ICCV 2003). Basically, graph cuts can be used to compute geodesics and minimal surfaces. Given a locally varying metric you can construct a graph so that min cuts on the graph are geodesics for that metric.

Level sets

- Curve evolution is a great idea — but what are the limitations? Self-intersections, or more generally topology changes, are the worst problem (there are also numerical issues, especially stopping conditions).

- There is a beautiful solution to these problems called *level sets*. Basic idea: instead of evolving the curve, we view the curve as the zero set of a surface and evolve the surface. The curve C is defined by $u(x, y) = 0$.

- Intead of evolving $C(s)$ as a function of time we evolve $u(x, y)$. The curve evolution equation $\frac{\partial C}{\partial t} = VN$ can be reformulated as the level set equation:
$$\frac{\partial u}{\partial t} = V||\nabla u||.$$
This updates every "pixel" of u.

- Amazing fact: if you update u according to this equation, *every* level set of u will update according to the curve evolution equation.