CS6630 Homework 3

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Problem 1. This homework deals with solutions for radiative transfer in an infinite layer of homogeneous medium—think of a plate of glass with scattering in the interior, or a cloud layer in the atmosphere. The layer is large enough in x and y that the radiance in and near the layer is independent of x and y, but it could be thin in the z direction. Consider such a layer of thickness d_0 that contains a medium with volume emission, absorption, and scattering coefficients ϵ, σ_a, and σ_s and phase function f_p. It is observed by a camera or other radiance detector from the outside. First suppose that $\sigma_a = \sigma_s = 0$.

a. Show that in the case where the refractive index inside and outside the layer is the same, the radiance emitted in direction ω is

$$L(\omega) = d_0\epsilon/\mu.$$

Infer the definition of μ. What uncomfortable prediction does this model make?

b. Suppose the layer is instead a piece of (glowing) glass ($\eta = 1.5$) surrounded by air, and the back side is painted perfectly black (no reflection from the back surface). Show that the radiance emitted in direction ω is

$$L(\omega) = \frac{T(\eta, \mu) d_0 \epsilon}{\eta^2 \mu'}.$$

Infer the definition of μ'.

c. Remove the black paint from the back; now you have to account for light reflected internally. Show that the radiance is now

$$L(\omega) = \frac{d_0 \epsilon}{\eta^2 \mu'}.$$

Now let $\sigma_a > 0$ but still keep $\sigma_s = 0$. From now on, disregard internal reflection at the interface, so we only have direct emission (and later, single scattering).

d. Show that the radiance emitting from this layer is

$$L_e(\omega) = T(\eta, \mu)(1 - \tau)\frac{\epsilon}{\eta^2 \sigma_a} + T(\eta, \mu)^2 \tau L_b$$

Where L_b is the radiance entering from the back and $\tau = \exp(-\sigma_a d_0/\mu')$. (The correction to account for internal reflection is again simple—but you don't need to include it in your writeup.)

e. What is the radiance emitted by a very thick layer?

Now replace the self-emitting medium with a more ordinary scattering medium, so $\epsilon = 0$, $\sigma_s > 0$. Suppose it is illuminated by parallel light coming from a direction ω_i, on the same side of the layer as the viewer, that would produce irradiance E_i on a surface facing the source. Ignore internal reflection.

f. Show that the irradiance available for scattering at depth d is

$$E'_i(d) = T(\eta, \mu_i)\frac{\mu_i}{\mu'_i}\exp(-\sigma_t d/\mu'_i)E_i.$$

Also define μ_i and μ'_i. (This formula gives scalar irradiance, or equivalently in this context, irradiance for a surface perpendicular to the illumination direction.)

g. Show that the radiance due to single scattering exiting the layer is

$$L_s(\omega_s) = \frac{T(\eta, \mu_i)T(\eta, \mu_s)}{\eta^2}\frac{\sigma_s}{\sigma_t}f_p(\omega'_i, \omega'_s)\frac{\mu_i}{\mu'_i + \mu'_s} \left(1 - \exp\left(-\frac{\mu'_i + \mu'_s}{\mu'_i\mu'_s}\sigma_t d_0\right)\right)E_i.$$

Provide definitions for the ωs and μs.

h. What is the radiance scattered by a very thick layer?

i. Work out the radiance when the source is on the opposite side of the layer from the viewer.

Problem 2. This problem is about diffusion-based solutions to scattering in an infinite homogeneous medium, accounting for multiple scattering as well as single scattering. Perhaps the most elementary problem to be solved is the radiance distribution due to a point source, which we conventionally place at the origin. Since everything is symmetric, we know the radiance distribution has to be spherically symmetric, so it can only depend on the distance from the source.

The classical diffusion approximation that we've looked at in class provides a simple approximate solution to this problem. We'll look at where it comes from and how accurate it is relative to an exact solution given below. The diffusion approximation reads:

$$D\nabla^2\phi(\mathbf{x}) = \sigma_a\phi(\mathbf{x}) - Q_0(\mathbf{x})$$

where D is called the diffusion constant and has the value $1/(3\sigma'_t)$. The source term Q_0 is zero everywhere except at the origin, and at the origin it's not defined because of the point source, so we'll avoid actually looking at radiance values at $\mathbf{x} = \mathbf{0}$.

a. Let us derive the solution using the time-honored "guess and plug in" method. Suppose we remember that the solution has the general form

$$\phi(\mathbf{x}) = a\frac{e^{-br}}{r}$$

where $r = \|\mathbf{x}\|$, for some positive real numbers a and b. By requiring that ϕ satisfies the diffusion equation above for all $\mathbf{x} \neq \mathbf{0}$, figure out what b must be. *Hint:* look up the formula for the Laplacian operator in spherical coordinates; otherwise you will want to use a symbolic algebra program to compute the derivatives.

b. Another conclusion of the diffusion approximation, in the absence of directional volume sources, is called Fick's law, which relates the scalar irradiance and the vector irradiance:

$$\vec{E}(\mathbf{x}) = -D\nabla\phi(\mathbf{x})$$

What is the vector irradiance at a point $\mathbf{x}$ in the volume ($\mathbf{x} \neq \mathbf{0}$)?

c. If we set $\sigma_a = 0$, then particles never go away; they bounce around, spreading farther and farther out in space. This means the net outward flux across any surface containing the source had better be equal to the power of the source. Find the net flux by integrating the vector irradiance and use the very simple resulting expressions to figure out the value of a. *Hint:* This is a very simple calculation.

If we only need to know about fluence, there is an exact analytical solution (not in closed form but easy to compute) available for the point source problem. It is found using a simple Fourier transform calculation that results in a messy integral for the inverse Fourier transform, with the result

$$\phi(\mathbf{x}) = \frac{\Phi\sigma_t}{4\pi\|x\|}\left[P_d \exp(-\sigma_t\|x\|/A) + \int_1^\infty g(t,\alpha)\exp(-\sigma_t\|x\|t)dt\right]$$

for a source of power Φ at the origin. Three subsidiary expressions, L, P_d, and g, are required:

$$2 = \alpha A \ln\left(\frac{A+1}{A-1}\right)$$

$$P_d = \frac{2(A^2-1)}{\alpha A^2(1 - A^2(1-\alpha))}$$

$$g(t,\alpha) = \left[\left(1 - \frac{\alpha}{t}\tanh^{-1}\left(\frac{1}{t}\right)\right)^2 + \left(\frac{\pi}{2}\frac{\alpha}{t}\right)^2\right]^{-1}$$

(Yes, finding A requires solving a root-finding problem.) This same solution can be found in the neutron transport literature (Case & Zweifel) or in the radiative transport literature (Ishimaru).

I have written up this solution in a Matlab function to save you the trouble of typing it in and setting up the two 1D numerical calculations required.

d. Plot your diffusion solution as a function of distance for the point-source problem against the exact solution, for various values of the medium parameters. You will need to figure out some way to scale the axes (often it is useful to plot some ratio or product of quantities to avoid graphing functions with singularities) so that you can see what is going on despite the wide dynamic range. Describe the set of medium parameters and distances for which the diffusion approximation seems to be doing a good job, and hand in plots that support your claim.