CS633 Spring 06 — Problems 1

Alan Demers

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These simple problems are taken almost verbatim from Chapter 1 of [L04]. I added a brief discussion of representing languages by monadic second-order formulas, since we didn't get to that in lecture on Thursday.

1 First Order Graph Properties

Show how to express the following graph properties in first-order logic:

a) A graph is complete (every vertex is connected by an edge to every other vertex).

b) A graph has at least two vertices of degree (*exactly*) 3.

c) Every vertex is connected by an edge to a vertex of degree (*exactly*) 3.

2 Existential Second Order Graph Properties

Show how to express the following graph properties in $\exists SO$ logic:

a) A graph has an independent set X of size at least k. (A set X of vertices is *independent* if no two vertices in X are connected by an edge.)

b) A graph on n vertices has an independent set of size at least $n/2$.

c) A graph has a kernel. (A set X of vertices is a *kernel* if X is an independent set and every vertex not in X is connected by an edge to some vertex in X.)

3 String Properties

Given a string $s = s_1 s_2 \ldots s_n$ over alphabet $\{a, b\}$, we create a structure M_s as follows: The universe is $\{1, 2, \ldots, n\}$, corresponding to positions in the string. There is a single binary relation $<$ whose interpretation is the usual order on the natural numbers. There are two unary relations A and B such that $A(i)$ is *true* if $s_i = a$ and $B(i)$ is *true* if if $s_i = b$.

We say sentence ϕ defines language $L \subseteq \{a, b\}^*$ if

$$M_s \models \phi \quad \iff \quad s \in L$$

We may allow ϕ to be a first-order sentence or an existential monadic second-order sentence.

a) Give a definition of the regular language

$$a^*(b+c)^*aa*$$

by a first-order sentence.

b) Give a definition of the regular language

$$(aaa)^*(bb)^+$$

by a monadic second-order sentence.

c) (a bit more difficult) Suppose you are given a finite automaton A (you may assume wlog that A is deterministic). Show how to construct a monadic existential second-order sentence ϕ_A that defines the language accepted by A according to

$$M_s \models \phi_A \quad \Longleftrightarrow \quad s \in L(A)$$

Note you may need quite a few second-order existential quantifiers. This is one direction of the claimed-in-lecture equivalence between monadic second-order and regular sets.