

CS 630: Learning from Clickthrough Data in IR

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(Many slides based on those of Thorsten Joachims)

Consider the Web Search Problem...

- Standard method: PageRank.
 - What are its assumptions?
 - When can't it work?
 - What else can't it do?
- Can we use Machine Learning to help?

Why Learn a Retrieval Function?

- Faster.
- Personalizable.
- Use of Feedback.
- Can optimize specific cost function.



The Problem

- Goal: "Put some relevant documents at the top of the list."
- Input: Query, Click data.
- Output: Ranking function.
- Method: Ranking SVM.

Query
"Support Vector Machine"

Ranking Presented to User

1. Kernel Machines
<http://svm.first.gmd.de/>
2. Support Vector Machine
<http://bolivar.freeservers.com/>
3. SVM-Light Support Vector Machine
<http://ais.gmd.de/~thorsten/svm-light/>
4. An Introduction to Support Vector Machines
<http://www.support-vector.net/>
5. Support Vector Machine and Kernel ... References
<http://svm.research.bell-labs.com/SVMrefs.html>
6. Archives of SUPPORT-VECTOR-MACHINES ...
<http://www.iscmail.ac.uk/lists/SUPPORT..>
7. Recent Technologies: SVM (some applet)
<http://svm.research.bell-labs.com/SVT/SVMsvt.html>
8. Royal Holloway Support Vector Machine
<http://svm.dvc.rhnc.ac.uk/>

How does this differ from...

- Classification?

ranking 1: - - + - - - - - - - -
ranking 2: - - - - - - - - - - +

⇒ Both return one relevant doc, with same error rate, but one is clearly better rank.

- Regression?

Regression (estimating a function) needs target values, we don't have them.

How do we Evaluate a Ranking?

- We could count the number of pair flips:

Ranking 1 | + - - + - - - - - - - - Cost: 2
Ranking 2 | - + + - - - - - - - - Cost: 2
Ranking 3 | + - + - - - - + - - - Cost: 6

- But its hard to optimize over this function.

What Does the Feedback Mean?

- Lets do a simple experiment.
- Let q be the query, d a document.
- Consider three ranking functions.
- Randomly pick a ranking function for each query.

Bool(q,d)	Boolean Retrieval function
TFIDF(q,d)	Weight query words by TFIDF score
HTML(q,d)	Also look at the HTML

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How do we compare two rankings?

- Consider two rankings. Which is better?
 - Are more clicks good or bad?
 - Is the rank of the clicks informative?

	Average Click Rank
Bool(q,d)	6.26
TFIDF(q,d)	6.18
HTML(q,d)	6.04

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Impact of Presentation Bias

Average Click Rank		Used for Computing Click Rank		
		Bool	TFIDF	HTML
P R E S E N T A T I O N	Bool	6.26	46.94	28.87
	TFIDF	54.02	6.18	13.76
	HTML	48.52	24.61	6.04

Boyan, Freitag, Joachims, 1996

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Impact of Presentation Bias

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A New Feedback Mechanism

- Assume “clicking” means it might be relevant.
- Assume “not clicking” means its probably not.
- By skipping over something then clicking below, the user is giving us a preference judgment.

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A New Feedback Mechanism

- We don't know about 1.
- 3 is more relevant than 2.
- 7 is more relevant than 2.
- 7 is more relevant than 4.
- 7 is more relevant than 5.
- 7 is more relevant than 6.
- We don't know about 8.

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A problem?

- Consider some query q .
- Assume a deterministic ranking function.
- All results are always in the same order.
- What possible constraints can we generate?
- What happens if the ranking is perfect?
- How do we fix it?

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Assumptions

- Assumption 1:
The user is more likely to click on something if it is relevant.
- Assumption 2:
There is no bias towards one retrieval strategy except relevance.

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Learning to Rank

- User says “ d_i better than d_j for query q ”

$$w \cdot \phi(q, d_i) \geq 1 + w \cdot \phi(q, d_j)$$

$$\Phi_{qij} = \phi(q, d_i) - \phi(q, d_j)$$

$$w \cdot \Phi_{qij} \geq 1 - \xi_a$$

- For each preference we get one constraint.
- We can then run the SVM optimization:

$$\min_{w, \xi} \frac{1}{2} |w| + C \sum \xi_a$$

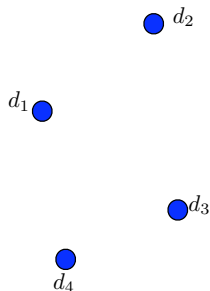
15

Typical features used

$$\phi(q, d) = \left[\begin{array}{l} \text{PageRank}(q, d) \\ \# \text{ of words of } q \text{ in title of } d \\ d \text{ ranked in top 10 of Google} \\ \text{Country code of URL is } X \\ \text{Cosine between URL and } q \\ d \text{ clicked on for } q \end{array} \right]$$

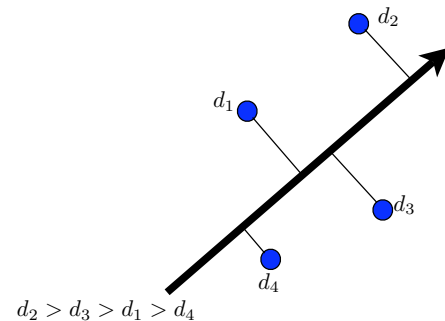
16

What does it look like?



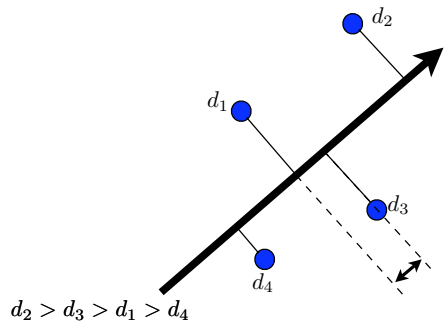
17

What does it look like?



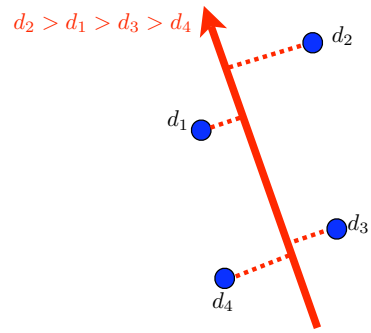
18

What does it look like?



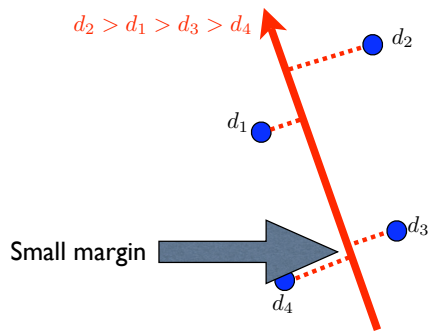
19

What does it look like?



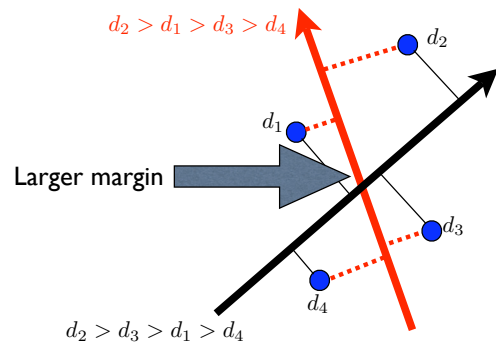
20

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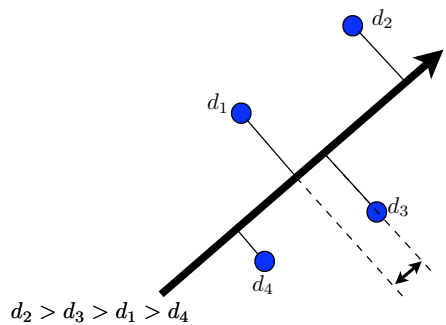
21

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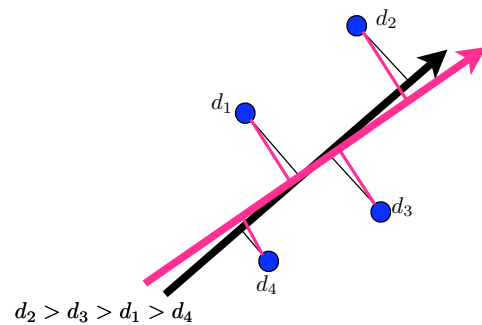
22

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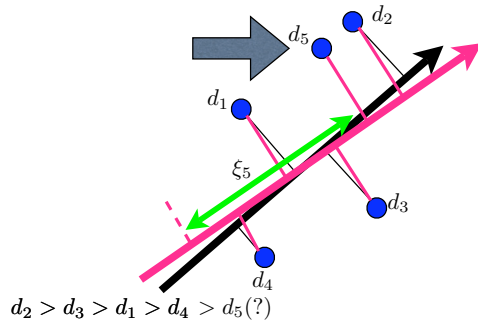
23

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What does it look like?



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Evaluation

- We have a new ranking function. Is it better?

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OLD Ranking
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Property of Combined Ranking

- Combine two rankings A and B so that user scanning from top to bottom always sees the same number of hits from each:

$$A = \{a_1, a_2, \dots\} \quad B = \{b_1, b_2, \dots\}$$

$$\{c_1, \dots, c_n\} = \{a_1, \dots, a_\alpha\} \cup \{b_\alpha, \dots, b_\beta\}$$

$$\text{Such that } \beta \leq \alpha \leq \beta + 1$$

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Specific Experiment

- Used a combination of current search engines.
- Displayed URL and title of documents.
- Asked users to use it for a while, recording use.
- Then learned weights for new retrieval function.
- Interleaved with other rankings to test results.

Joachims, 2002

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Feature Weights Learned

Weight	Feature
0.60	Cosine between query and abstract
0.48	Ranked in top 10 by Google
0.24	Cosine between query and URL
0.24	Ranked top 1 by exactly one SE
	...
0.17	Country code of URL is "de"
0.16	Ranked top 1 by HotBot
	...
-0.15	Country code of URL is "fi"
-0.17	Length of URL in characters
-0.32	Not in top 10 in any SE
-0.38	Not top 1 for any SE

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Evaluation

Ranking A	Ranking B	A Better	B Better	Tie	Total
Learned	Google	29	13	27	69
Learned	MSN Search	18	4	7	29
Learned	Toprank	21	9	11	41

- Are these results significant?

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Are these significant?

- How likely are we to say A is better 29 out of 42 times they differ, B is better 13 times?
- Use McNemar's Test (Paired t-test):

$$P(N_A > n_A | p, n) = \sum_{i=0}^{n_A} \frac{n!}{i!(n-i)!} p^i$$

Result: Learned Ranking is Better

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What about the Assumptions?

- Assumption 1:

The user is more likely to click on something if it is relevant.

$$P(C|R) = P(R|C) \frac{P(C)}{P(R)} \approx \frac{C_R}{C} \frac{C}{R}$$

$$P(C|N) = P(N|C) \frac{P(C)}{P(N)} \approx \frac{C_N}{C} \frac{C}{N}$$

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What about the Assumptions?

- Assumption 2:

There is no bias towards one retrieval strategy except relevance.

$$E[C_{R,A}] = E\left[C_R \frac{R_A}{R}\right] \quad E[C_{N,A}] = E\left[C_N \frac{N_A}{N}\right]$$

$$E[C_{R,B}] = E\left[C_R \frac{R_B}{R}\right] \quad E[C_{N,B}] = E\left[C_N \frac{N_B}{N}\right]$$

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