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Last time: $\mathrm{NCUT}(S) = \frac{\mathrm{cut}(S)}{v(S)} + \frac{\mathrm{cut}(\bar{S})}{v(\bar{S})}$

$\mathrm{cut}(S) = \sum_{i\in S, j\in \bar{S}} A_{ij} \qquad v(S) = \sum_{i \in S} d_i$

$\phi(S) = \mathrm{cut}(S) / \min(v(S), v(\bar{S}))$

$$\frac{\mathrm{cut}(S)}{v(S)} + \frac{\mathrm{cut}(\bar{S})}{v(\bar{S})} \le \frac{\mathrm{cut}(S)}{\min(v(S), v(\bar{S}))} + \frac{\mathrm{cut}(\bar{S})}{\min(v(S), v(\bar{S}))}$$

$$= 2\phi(S)$$

$$\ge \frac{\mathrm{cut}(S)}{\min(v(S), v(\bar{S}))} = \phi(S)$$

$\lambda_2 \le \min_S \mathrm{NCUT}(S)$

Cheeger

$\lambda_2/2 \le \min_S \phi(S) \le \sqrt{2\lambda_2}$

$$\min_x \; x^T L x \quad \text{s.t.} \quad 1^T D x = 0 \quad x^T D x = 1$$

$$z = D^{1/2} x \qquad \min \; z^T N z \qquad N = D^{-1/2} L D^{-1/2}$$

$$\text{s.t.} \quad 1^T D^{1/2} z = 0 \quad z^T z = 1$$

$$N z_* = \lambda_2 z_* \implies z_*^T N z_* = \lambda_2$$

$$\lambda_2 = \min_{1^T D^{1/2} z = 0} \frac{z^T N z}{z^T z} \quad \text{at } z_* \qquad \leq \min_S \mathrm{NCUT}(S)$$

$$= \min_{1^T D x = 0} \frac{x^T L x}{x^T D x} \quad \left(\text{at } x^* = D^{-1/2} z_*\right)$$

Let $\underline{\mathbb{1}^T D y} = 0 \qquad S_t = \{ i \mid y_i < t \}$

$\boxed{\text{Claim}}$: $\quad \min_t \phi(S_t) \leq \sqrt{2 \frac{y^T L y}{y^T D y}} \stackrel{\Delta}{=} \sqrt{2R}$

$\min_y \frac{y^T L y}{y^T D y} \Rightarrow \lambda_2(N) \quad y_* = D^{-1/2} z_* \quad N z_* = \lambda_2 z_*$

WLOG: $y_1 \leq \cdots \leq y_n$

$c = \min_k \sum_{i=1}^{k} d_i \geq v(V)/2$