CS 624: Numerical Solution of Differential Equations
Spring 2004
Problem Set 5

Handed out: Wed., Apr. 7.

Due: Fri., Apr. 16 in lecture.

1. Consider the Lax-Friedrichs method for a scalar conservation law $u_t = -(f(u))_x$. Show that Lax-Friedrichs is actually second-order for a modified PDE $u_t = -(f(u))_x + T$, where T is a sum of some other terms that you must determine, all of which should be $O(k)$. The formula for T will involve h and k. In your analysis, assume $k/h = \lambda$, where λ is a fixed constant.

   If you carry this out correctly, T will have three terms, two of which are multiples u_{xx}. If the two latter terms are added to arrive at a single term Au_{xx}, is the coefficient A positive or negative? Take into account the CFL condition.

   [Hint: Plug in the exact solution and carry out a Taylor series expansion. Figure out how to define u_t (i.e., figure out what T has to be) in such a way that the ku_t term on the left-hand side of the Taylor expansion can cancel the $O(k^2)$ term on the left-hand side and the $O(h^2)$ term on the right-hand side.]

2. (a) Show that the viscous Burgers equation $u_t + uu_x = au_{xx}$, where $a > 0$ is a constant, can be transformed analytically to a diffusion equation.

   [Hint: Use the "Cole-Hopf transformation": Show that if v satisfies $v_t + v_x^2/2 = av_{xx}$, then $u = v_x$ is a Burgers solution. Then substitute $v = -2a \log w$, coming up with a new equation for w.]

   (b) Note that the diffusion equation $w_t = aw_{xx}$ defined on $[0, 2\pi] \times [0, \infty)$ with periodic BC's (i.e., $w(0, t) = w(2\pi, t)$ for all $t \geq 0$) has as an exact analytic solution $w(x, t) = C + \exp(-n^2 at) \sin(nx)$ for any integer n and real constant C. Figure out the corresponding Burgers' analytic solution.

3. Write down a semi-discretization of viscous Burgers' equation, that is, discretize in space only yielding a system of ODE's. Your discretization should be "consistent" in the intuitive sense that it is made up of terms that correspond to finite difference approximations to the terms in Burgers equation, but you do not need to formally define or prove that it is consistent. Your system of ODE's should be based on spatial domain $[0, 2\pi]$ including periodic boundary conditions.

4. Implement (in Matlab) the numerical method for the viscous Burgers equation in the last question. Use periodic boundary conditions ($u(0, t) = u(2\pi, t)$ for all $t \geq 0$). Use examples from 2(b) for initial conditions. As usual, you can use ODE15s to solve the ODEs.

Now compare (experimentally) your computed solution at time $t = 1$ to the exact analytic solution and try to determine (experimentally) the order of your method (order with respect to h). This may require some adjusting of the tolerances to ODE15s so that the role of k is not an issue.

Hand in listings of all m-files, at least two interesting plots, and a paragraph of conclusions.