CS 624: Numerical Solution of Differential Equations
Spring 2004
Problem Set 1

Handed out: Mon., Feb. 2.

Due: Wed., Feb. 11 in lecture.

1. Exercise 1.1.4(c) from Trefethen. Hint: The first two solutions you find should both have the form $u(t) = Cu^\beta$ for some constants C and β. The infinite family of solutions is obtained by gluing these two solutions together.

2. Consider two LMS methods in standard form given by (1.2.11) in the text. Suppose their orders of accuracy are p_1, p_2. WLOG, assume they have the same value of s. Their sum, suitably rescaled to ensure that the leading α_j is 1, is also a LMS method. Let p be the order of the resulting LMS method.

   (a) Show that $p \geq \min(p_1, p_2)$.

   (b) Show that if $p_1 \neq p_2$, then the inequality in (a) is tight.

   (c) Give an example to show that if $p_1 = p_2$, then the inequality in (a) is not necessarily tight. [Hint: there are two well known methods with $p_1 = p_2 = 1$ such that the order p of their sum is 2.]

3. Let
$$(x_1, y_1), \ldots, (x_n, y_n), (w_1, z_1)$$
be a sequence of $n+1$ real points in the plane such that $x_1 < x_2 < \cdots < x_n < w_1$. Assume $n > 0$. Show that there exists a unique polynomial p of degree at most n such that $p(x_i) = y_i$ for $i = 1, \ldots, n$ and $p'(w_1) = z_1$. Note: this theorem is used to establish the validity of the interpolation-based definition of the BDF family.

   [Hint: As in lecture, first show uniqueness, then call upon linear algebra to conclude existence. To show uniqueness, first argue that p' is uniquely determined. In the uniqueness proof, Rolle's theorem will help you find $n-1$ roots of the derivative of $p-q$, and there is already another root given.]

4. Consider a frictionless pendulum, whose equations of motion are:
$$\begin{aligned} \frac{d\theta}{dt} &= v, \\ \frac{dv}{dt} &= -\sin\theta \end{aligned}$$
where θ is the angle made by the pendulum with respect to vertical and v is the angular velocity of the pendulum. It can be verified mathematically that this system conserves energy, where energy is $-\cos\theta + v^2/2$.

Choose starting data $\theta_0 = \pi/2$ and $v_0 = 0$ (i.e., the pendulum is stationary and horizontal). Implement AB1 and AB2 for this system in Matlab and track the energy of the system. How well do each of them conserve energy for a fixed time step and fixed interval of integration? What happens to each when the time step is halved (but the interval of integration is fixed)?

Run AB1 for a very large number of steps. You will notice there is eventually a qualitative transition to a different kind of behavior. Can you explain this transition? (The same thing will happen to AB2, if the number of steps is large enough.)

Turn in listings of your m-files, a paragraph or two of conclusions and at least one interesting plot.