CS 624: Numerical Solution of Differential Equations
Spring 2004
**Prelim 2**

Handed out: Tues., Mar. 30.

This exam has four questions. The questions are weighted equally. It counts for 20% of your final course grade (same as Prelim 1). This exam is due back at the end of lecture Friday, April 2 if you picked up the exam on Tuesday, March 30 or at the end of lecture on Monday, April 5 if you picked up the exam on Friday, April 2.

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than Trefethen then you must cite them.

**Academic integrity.** You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until Tuesday, April 6. The following kind of cooperation is allowed: You are permitted to borrow and photocopy someone else's lecture notes or other publicly-available course material. Please write and sign the following statement on your solutions: "I have neither given nor received unpermitted assistance on this exam."

You are not allowed to send any email or otherwise make any on-line posting concerning the questions on this exam until after it is over. But you are allowed to consult publicly-available websites and search engines.

**Help from the instructor.** The only help available will be clarification of the questions. No help will be given towards finding a solution.

**Late acceptance policy.** Solutions turned in after lecture but before 5:00 p.m. on the due date will be accepted with a 10% penalty. The full late penalty is applied even if a portion of the solutions are handed in on time.

1. Recall the pendulum problem from Problem Set 1, Question 4. Consider changing the pendulum to an exactly energy conserving system by replacing one of the two equations with a constraint expressing conservation of energy, thus yielding a system of differential algebraic equations (as in PS 3, Question 4).

   Suppose the initial energy $E_0$ satisfies $E_0 < 1$. Explain in detail why the the DAE system will run into the same difficulty as was encountered in PS 3, Question 4, namely, the index does not remain one throughout the trajectory. On the other hand, if $E_0 > 1$, then explain why it is possible to replace one differential equation with a constraint and obtain a truly index-one DAE.

2. Consider applying the Euler method for $u_t = u_{xx}$ on the finite spatial domain $[0, 1]$. Suppose the boundary condition at the left end of the domain (i.e., at $x = 0$) is $u_x = 0$.

   (a) Two ways to implement this boundary condition for the Euler method are as follows: define $v^n_{-1}$ to be identically equal to $v^n_0$ for all $n$, or define $v^n_{-1}$ to be identically equal to $v^n_1$ for all $n$. Write down the Euler method formula for $v^{n+1}_j$ in the special case $j = 0$ after applying each of these implementations of the boundary condition.

(b) What is the error accrued in one step of computing $v_0^n$ after using either of the formulas in part (a)? (Use the approach from lecture, namely, plug the exact solution into the difference formula, take a formal Taylor series expansion, and apply the PDE as well as the boundary condition to simplify.) Write your answers in terms of $k$ alone by making a reasonable assumption about the relationship between $k$ and $h$.

3. Consider all possible methods for $u_t = u_x$ of the form

$$v_j^{n+1} = v_j^n + \frac{k}{h}\left(\beta_{-1}v_{j-1}^n + \beta_0 v_j^n + \beta_1 v_{j+1}^n\right)$$

where $\beta_{-1}, \beta_0, \beta_1$ are coefficients that define the method. Assume $k = \lambda h$, where $\lambda$ is a fixed constant. The coefficients $\beta_{-1}, \beta_0, \beta_1$ may depend on $\lambda$. For example, the upwind, downwind, and Lax-Wendroff methods all have this form.

Determine all choices of $\beta_{-1}, \beta_0, \beta_1$ that yield consistent methods. Determine all choices that yield second-order methods or higher. Confirm that the Lax-Wendroff method satisfies your conditions for second-order or higher. Could any other consistent choices of the coefficients yield second order or higher? The answer to this latter question may depend on $\lambda$.

4. Consider a consistent explicit one-step finite difference method for the diffusion equation of the form

$$v_j^{n+1} = v_j^n + \frac{k}{h^2}\sum_{m=-r}^{r}\alpha_m v_{j+m}^n$$

where $\alpha_{-r}\ldots,\alpha_r$ are constant coefficients (independent of $n$ and $j$; also independent of $k$ and $h$). For example, the Euler method presented in lecture has this form. Show that for this method to be stable, the time step must satisfy $k \le C_r h^2$, where $C_r$ depends on $r$.

Hint: A consistent formula cannot have all the $\alpha_m$'s equal to zero. Use the consistency requirement to demonstrate the inequality $\sum_{m=-r}^{r}|\alpha_m|^2 \ge C_r'$ where $C_r'$ is a positive constant that you will find and that depends on $r$. Then use this minimum value and (2.2.8) of the text to establish a minimum possible value $C_r''$ for

$$\int_{-\pi/h}^{\pi/h}\left|\sum_{m=-r}^{r}\alpha_m e^{im\xi h}\right|^2 d\xi$$

Then use this minimum possible value to establish a lower bound $C_r'''$ on

$$\max_{\xi\in[-\pi/h,\pi/h]}\left|\sum_{m=-r}^{r}\alpha_m e^{im\xi h}\right|.$$

Finally, use this lower bound to conclude the proof.