CS 624: Numerical Solution of Differential Equations
Spring 2004
Prelim 1

Handed out: Tues., Feb. 24.
This prelim has five questions worth 75 points total. It is closed book and closed note, but you may use an 8.5×11 sheet of paper written with notes on both sides that you have prepared in advance.

1. One way to initialize an s-step LMS method, $s > 1$, is to define $v^j = u(0) + jku'(0)$ $j = 0, \ldots, s-1$ for the IVP $du/dt = f(u,t)$, $u(0) = u_0$. Is this a good approach for initialization? In particular, explain what impact this initialization has on the computed solution $v^{1/k}$, which is supposed to approximate $u(1)$. Take into account the Dahlquist theorem and the method's order of accuracy.

2. Consider all possible LMS methods of the form $v^{n+1} + \alpha_1 v^n + \alpha_0 v^{n-1} = \beta_1 f^n$. Determine conditions on $\alpha_1, \alpha_0, \beta_1 \in \mathbf{R}$ to ensure that this method is consistent (first order or greater). Determine conditions that ensure that this method is second order or greater. Finally, among all the choices of coefficients that make this method consistent, which ones yield a D-stable formula?

3. Let $\mathbf{u}, \mathbf{v}$ be two N-vectors, each functions of t, and consider solving the differential-algebraic system $d\mathbf{u}/dt = g(\mathbf{u}, \mathbf{v})$ and $B\mathbf{v} = \mathbf{b}$ where g is a function $\mathbf{R}^{2N} \to \mathbf{R}^N$, B is a given $N \times N$ matrix, and $\mathbf{b}$ is a given N-vector. One way to solve this problem that does not involve computing $B^{-1}\mathbf{b}$ is to try to make $\mathbf{v}$ always satisfy the linear equation by penalizing any departure from $B\mathbf{v} = \mathbf{b}$. For example, the following ODE for $\mathbf{v}$ has this property: $d\mathbf{v}/dt = MB^T(\mathbf{b} - B\mathbf{v})$, where $M \gg 0$ is a penalty parameter. (Thus, the original DAE is transformed into an IVP involving the original differential equation for $\mathbf{u}$ and this new one for $\mathbf{v}$.) Explain the major drawback with this approach. [Hint: consider the eigenvalues of the Jacobian and stiffness considerations. Note that the set of eigenvalues of a block upper triangular matrix is the union of the eigenvalues of the individual diagonal blocks.]

4. Consider the IVP $du/dt = au^p$, $u(0) = 0$ where $0 < p < 1$ and $a > 0$ is a scalar. (a) Find all real solutions to this IVP of the form $u(t) = bt^q$. (b) Suppose the BE method is applied to this IVP. For which cases does the BE equation, when it is solved to obtain v^1, have more than one solution?

5. Recall that AB2 is defined according to the rule:

$$v^{n+1} - v^n = \int_{t_n}^{t_n+1} q(t)\,dt$$

where $q(t)$ is the linear interpolant of (t_{n-1}, f^{n-1}), (t_n, f^n). Write this formula out with explicit values of the coefficients for the special case that the time step sizes alternate k, $2k$, k, $2k$, etc., i.e., for the case that $t_n - t_{n-1} = k$ if n is odd else $t_n - t_{n-1} = 2k$.