Theory for the randomized SVD

Recall some notation from the previous lecture: given \( A \in \mathbb{R}^{m \times n} \), we defined
\[
Y = A\Omega, \quad \Omega \in \mathbb{R}^{n \times (k+p)}, \quad (\Omega)_{ij} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1).
\]
and denoted \( P_Y \) for the spectral projector to the range of \( Y \). Moreover, recall that we denoted \( \Omega_1 := V_1^\top \Omega, \quad \Omega_2 := V_2^\top \Omega, \)
where \( V_1, V_2 \) are the right singular vectors of \( A \). Notice that because \( V_1, V_2 \) are orthogonal to each other, \( \Omega_1 \) and \( \Omega_2 \) are independent; moreover, since \( V_1, V_2 \) have orthonormal columns, \( \Omega_1 \) and \( \Omega_2 \) have i.i.d. \( \mathcal{N}(0, 1) \) elements as well.

We previously showed the following Lemma:

**Lemma 1.** Assuming that \( \Omega \) has full row rank, it holds that
\[
\|(I - P_Y)A\|^2_2 \leq \|\Sigma_2\|^2_2 + \|\Sigma_2\Omega_2\Omega_1^+\|^2_2,
\]
where \( X^+ \) denotes the pseudoinverse of the matrix \( X \).

To move forward, we need to understand \( \Omega_2\Omega_1^+ \) when \( \Omega \) has i.i.d. standard normal entries. In that case, \( V_1^\top \Omega \) and \( V_2^\top \Omega \) also consist of i.i.d. Gaussian elements, by properties of orthogonal transforms of Gaussians. This would also hold for any choice of so-called isotropic distribution; for example, we could have sampled the columns of \( \Omega \) from Rademacher random vectors. See [Vershynin, 2018] Chapter 3 for a discussion about isotropic random vectors.

We now introduce some technical results necessary for the remainder of the proof:

**Proposition 2.** Fix \( S, T \) and draw a sample \( G \) with \( G_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1) \). Then:
\[
\left( \mathbb{E} \left[ \|SGT\|_F^2 \right] \right)^{1/2} = \|S\|_F \|T\|_F, \quad \mathbb{E} \left[ \|SGT\|_2 \right] \leq \|S\|_2 \|T\|_F + \|S\|_F \|T\|_2.
\]

**Theorem 3.** Let \( G \in \mathbb{R}^{k \times (k+p)} \) have i.i.d. \( \mathcal{N}(0, 1) \) entries. Then its pseudoinverse satisfies:
\[
\left( \mathbb{E} \left[ \|G^+\|_F^2 \right] \right)^{1/2} = \sqrt{\frac{k}{p-1}}, \quad \mathbb{E} \left[ \|G^+\|_2 \right] \leq \frac{e\sqrt{k+p}}{p}.
\]

The following theorem about Lipschitz concentration is standard ([Vershynin, 2018] Chapter 5):
Theorem 4. Let \( h : \mathbb{R}^{m_1 \times m_2} \to \mathbb{R} \) satisfy \( |h(X) - h(Y)| \leq L \|X - Y\|_F, \forall X, Y \). Then, if \( G \) is a Gaussian random matrix, we have

\[
P(h(G) \geq \mathbb{E}[h(G)] + Lt) \leq \exp\left\{ -\frac{t^2}{2} \right\}. \tag{8}
\]

Finally, we need a tail bound for the Frobenius and spectral norms of the pseudoinverse:

Proposition 5. For \( G \in \mathbb{R}^{k \times (k+p)} \) with \( G_{ij} \sim \mathcal{N}(0, 1) \) and \( p \geq 4 \), we have

\[
P(\|G^+\|_F \geq \frac{3k}{p+1} - t) \leq t^{-p}, \forall t \geq 1 \tag{9}
\]

\[
P(\|G^+\|_2 \geq \frac{\sqrt{k+p}}{p+1} t) \leq t^{-(p+1)}, \forall t \geq 1 \tag{10}
\]

We are now in good shape to prove the desired statement. In fact, we will prove something stronger, as shown below:

Theorem 6. For \( Y \) defined as in (1), we have

\[
\mathbb{E}[\|(I - P_Y)A\|_2] \leq \left(1 + \sqrt{\frac{k}{p-1}} + \frac{e\sqrt{k+p}}{p} \sum_{j>k} \sigma_j^2 \right)^{1/2} \tag{11}
\]

where \( sr(B) \) denotes the stable rank of \( B \), defined as

\[
sr(B) = \frac{\|B\|_F^2}{\|B\|_2^2} = \sum_{i=1}^n \left( \frac{\sigma_i}{\sigma_1} \right)^2 \tag{12}
\]

Proof. Starting from our deterministic bound, we apply the expectation operator and obtain

\[
\mathbb{E}[\|(I - P_Y)A\|_2] \leq \mathbb{E}\left[\left( \sigma_{k+1}^2 + \|\Sigma_2\Omega_2^+\|_2^2 \right)^{1/2} \right] \leq \sigma_{k+1} + \mathbb{E}\left[\|\Sigma_2\Omega_2^+\|_2^2 \right] \tag{13}
\]

where (13) is simply \( \sqrt{x+y} \leq \sqrt{x} + \sqrt{y} \). We proceed by analyzing the second term above. We use the tower property of expectation and independence of \( \Omega_1 \) and \( \Omega_2 \) to write

\[
\mathbb{E}[\|\Sigma_2\Omega_2^+\|_2^2] = \mathbb{E}_{\Omega_1}\left[ \mathbb{E}[\|\Sigma_2\Omega_2^+\|_2^2 | \Omega_1] \right] \leq \|\Sigma_2\|_2 \mathbb{E}[\|\Omega_1^+\|_F^2] + \|\Sigma_2\|_F \mathbb{E}[\|\Omega_1^+\|_2^2] \tag{14}
\]

where (14) follows from Proposition 2. Applying Theorem 3 to upper bound the expectation yields the claim.

Remark 1. The stable rank of a matrix is intimately related to the concept of statistical dimension or stable dimension. In particular, the algebraic dimension of a mathematical object may change abruptly even under small perturbations (e.g. adding gaussian noise \( \varepsilon Z \) to a low rank matrix), but the stable dimension changes smoothly as a function of the perturbation. An excellent treatment is available in (Vershynin, 2018, Chapter 7).

We can also prove a corresponding high probability statement.
Theorem 7. For all \( t, u \geq 1 \), we have
\[
\| (I - P_Y) A \|_2 \leq \left[ 1 + \sqrt{\frac{3k}{p+1}} t + t \frac{e \sqrt{k + p}}{p+1} \frac{\sigma_k}{\sqrt{2}} \frac{1}{\sqrt{2}} \| \Sigma_2 \|_2 \right] \cdot \sigma_{k+1} + ut \frac{e \sqrt{k + p}}{p+1} \sigma_{k+1} \tag{15}
\]
with probability at least \( 1 - 2t^{-p} - e^{-2t} \). In particular, setting \( t = p \) and \( u = \sqrt{2p \log p} \) gives probability of success at least \( 1 - 3p^{-p} \), as stated in the previous lecture.

Proof. The outline of the proof technique here is as follows: we have a “bad” event \( B \) that we want to control (i.e. \( (I - P_Y) A \|_2 \) being large), which depends on a deterministic quantity \( \| \Sigma_2 \|_2 \) and a random quantity \( \| \Sigma_2 \|_2 \Omega_1^+ \|_2 \). To control the latter, we can see that \( \Omega_1^+ \) is “well-behaved” (call this event \( E_1 \)); then, for a convenient choice of parameters, it is easy to see that the contrary happens with very small probability. If \( B \) denotes the “bad” event where \( \| \Sigma_2 \|_2 \Omega_1^+ \|_2 \) is not small, observe that
\[
P(B) = P(B \cap E) + P(B \cap E^C) = P(E) \cdot P(E) + P(B \mid E^C) \cdot P(E^C). \tag{16}
\]

The remainder of the proof is devoted to control of the above.

- for any \( t \geq 1 \), we define the event \( E_t \) as
\[
E_t := \left\{ \| \Omega_1^+ \|_2 \leq \frac{e \sqrt{k + p}}{p+1} t \text{ and } \| \Omega_1^+ \|_F \leq \sqrt{\frac{3k}{p+1}} \right\}. \tag{17}
\]

We can then see that
\[
P(E_t^c) = P\left( \left\{ \| \Omega_1^+ \|_2 \geq \frac{e \sqrt{k + p}}{p+1} t \right\} \cup \left\{ \| \Omega_1^+ \|_F \geq \sqrt{\frac{3k}{p+1}} \right\} \right) \tag{18}
\]
\[
\begin{align*}
\leq & P\left( \left\{ \| \Omega_1^+ \|_2 \geq \frac{e \sqrt{k + p}}{p+1} t \right\} \right) + P\left( \| \Omega_1^+ \|_F \geq \sqrt{\frac{3k}{p+1}} \right) \tag{19} \\
\leq & t^{-p} + t^{-(p+1)} \leq 2t^{-p}, \tag{20}
\end{align*}
\]

where \((*)\) is simply the union bound, and \((\sharp)\) follows by Proposition[5]

- let \( h(X) := \| \Sigma_2 X \Omega_1^+ \|_2 \). Then it follows that
\[
| h(X) - h(Y) | \leq \| \Sigma_2 \|_2 \| \Omega_1^+ \|_2 \| X - Y \|_F,
\]
i.e. \( h \) is Lipschitz with \( L := \| \Sigma_2 \|_2 \| \Omega_1^+ \|_2 \). This is true since
\[
| h(X) - h(Y) | \leq \| \Sigma_2 X \Omega_1^+ \|_2 - \| \Sigma_2 Y \Omega_1^+ \|_2 \leq \| \Sigma_2 X \Omega_1^+ - \Sigma_2 Y \Omega_1^+ \|_2 \leq \| \Sigma_2 \|_2 \| \Omega_1^+ \|_2 \| X - Y \|_2 \leq \| \Sigma_2 \|_2 \| \Omega_1^+ \|_2 \| X - Y \|_F,
\]
where \((\sharp)\) is just the (reverse) triangle inequality for norms, the penultimate inequality follows from the submultiplicative property of the spectral norm and the last inequality is \( \| A \|_2 \leq \| A \|_F \).
We can now apply Theorem 4 conditionally on $\mathcal{E}_t$ for $h(X)$ defined as above; using Eq. (14), we obtain:

$$
\mathbb{P}\left( h(X) \geq \|\Sigma_2\|_2 \sqrt{\frac{3k}{p+1}} t + \|\Sigma_2\|_F \frac{e\sqrt{k+p}}{p+1} t + \|\Sigma_2\|_2 \frac{e\sqrt{k+p}}{p+1} tu \Big| \mathcal{E}_t \right)
\leq \mathbb{P}\left( h(X) \geq \mathbb{E}[h(X)] + \|\Sigma_2\|_2 \|\Omega^\top\|_2 u \mid \mathcal{E}_t \right) \leq \exp \left\{ -\frac{u^2}{2} \right\},
$$

where we used the fact that if $\beta \leq \alpha$, then $\mathbb{P}(X \geq \alpha) \leq \mathbb{P}(X \geq \beta)$. Combining with the bound for $\mathbb{P}(\mathcal{E}_t)$ gives probability of failure at most $2t^{-p} + \exp \left\{ -\frac{u^2}{2} \right\}$, completing the proof after substituting the bound for $h(X)$ into the deterministic expression (3).

**Extensions to other random matrices**

It is possible to extend the above analysis to other types of random matrices, including structured random matrices enabling quick application. At the cost of an extra $\log k$ term, one can use the Subsampled Random Fourier Transform (SRFT) from Woolfe et al. (2008), which is defined as

$$
\Omega := \sqrt{\frac{n}{\ell}} DFR^\top,
$$

with $D = \text{diag}(d_1, \ldots, d_n)$ and $d_i$ are uniformly sampled from the unit complex circle. If we wish to remain in the real world, the Subsampled Random Hadamard Transform (SRHT) replaces $D$ with random signs and $F$ with the $n \times n$ Hadamard matrix $H_n$ (assuming for simplicity that $n = 2^p$ for some $p$). Both $F$ and $H$ can be applied from the right in time $O(mn \log n)$; the former is possible via the FFT, while the latter is via the Fast Walsh-Hadamard Transform. The analysis for the SRHT appears in Tropp (2011); see also Drineas and Mahoney (2018). The aforementioned structured random matrices find applications in fields extending beyond numerical linear algebra; the SRHT has been used as a design matrix in compressed sensing Do et al. (2011) and adjacent fields such as phase retrieval Duchi and Ruan (2019).

Here, we show how to adapt Theorem 7 to the SRHT sketching matrix. First, we need the following theorem from Tropp (2011):

**Theorem 8 (SRHT guarantees).** Let $r$ be the target rank, and draw an $n \times \ell$ SRHT matrix $\Omega$, where $\ell$ satisfies

$$
n \geq \ell \gtrsim \sqrt{r \log n}.
$$

Then, for a fixed orthonormal matrix $V$, it holds that

$$
\mathbb{P}\left( \{ \sigma_1(V^\top \Omega) \leq 1.48 \} \cap \{ \sigma_r(V^\top \Omega) \geq 0.40 \} \right) \geq 1 - c_1 r^{-1}
$$

for a universal constant $c_1 > 0$.

Then it suffices to note that for any $A$, it holds that $\|A^+\|_2 = \sigma^{-1}_{\text{min}}(A)$, so $\sigma_r(V^\top \Omega) \geq 0.40$ implies that $\|\Omega^\top\|_2 \leq \frac{1}{4\sigma_r}$. Similarly, $\|\Omega_2\|_2 \leq 1.48$, and combining with the deterministic bound from (3) yields the proof for SRHT matrices.

**References**


