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Problem: $Ax = b$ $\left(\min_x \|Ax - b\|_2^2 \text{ and } Ax = \lambda x \text{ later} \right)$

$\underset{n}{} \underbrace{[A]}_{n\times n} \underset{1}{[x]} = [b]$

Iterative solvers: only have access to Ax, $A^T x$ (not A_{ij})

Idea: search for solution in a subspace ($\subseteq \mathbb{R}^n$)

Krylov subspace: $K_k(A, b) = \operatorname{span}(\{b, Ab, A^2 b, A^3 b, \dots, A^{k-1} b\})$

Method: $\min_x \|Ax - b\|$ s.t. $x \in K_k(A, b)$ $\quad n\,[A]^{n}$ $\quad n\,[b]^{1}$

Overall plan:

$\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$

$\tilde{x}_j \in K_j(A, b)$

$\tilde{x}_{j+1}$ from $\tilde{x}_j$

Basis: $M = [b, Ab, \dots, A^{k-1} b]$

$\min_z \|\underbrace{AM}z - b\|$

write $x = Mz$ $\quad \|Fz - b\|_2 \Rightarrow$ least squares (MINRES)

$n\,[\;]^{k}\,{}_{k}[\;]^{1} - [\;]$

Approx. theory

$$K_k = \text{span}\{A^0 b, A^1 b, \dots, A^{k-1} b\}$$

$$x \in K_k \Rightarrow x = c_0 A^0 b + c_1 A^1 b + \dots + c_{k-1} A^{k-1} b$$
$$= p(A) b$$

$$\min_x \|Ax - b\| \quad \text{s.t. } x \in K_k(A, b) \iff \min_p \|p(A) b - b\| \quad \text{s.t. } p \in P_{k-1}$$

$$A = V \Lambda V^{-1} \qquad p(A) = V p(\Lambda) V^{-1}$$

$$\min \|q(\Lambda)\| \quad \text{s.t. } q \in P_{k-1}, \ q(0) = -1$$

$$\Rightarrow \min_{p \in P_{k-1}}$$
$$= \|V(p(\Lambda) V^{-1} b - V^{-1} b)\|$$
$$= \|V(p(\Lambda) - I) V^{-1} b\|$$
$$\le \|V\| \|V^{-1}\| \underbrace{\|b\|}_{} \underbrace{\|p(\Lambda) - I\|}_{}$$

RHS