

Nov 11, 2020

Problem: $Ax = b$ ($\min_x \|Ax - b\|_2^2$ and $Ax = b$ later)

$$\boxed{A} \cdot \boxed{x} = \boxed{b}$$

Iterative solvers: only have access to Ax , $A^T x$ (not A_{ij})

Idea: search for solution in a subspace ($\subseteq \mathbb{R}^n$)

Krylov subspace: $K_k(A, b) = \text{span}(\{b, Ab, A^2b, A^3b, \dots, A^{k-1}b\})$

Method: $\min_x \|Ax - b\|$ $\cdot \boxed{A} \cdot \boxed{b}$
s.t. $x \in K_k(A, b)$

Basis: $M = [b, Ab, \dots, A^{k-1}b]$

$$\min_z \|AMz - b\|$$

write $x = Mz$ $\cdot \begin{bmatrix} \boxed{A} \\ \boxed{M} \end{bmatrix} \cdot \boxed{z} - \boxed{b}$ \Rightarrow least squares (MINRES)

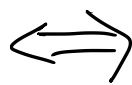
Overall plan:
 $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$
 $\tilde{x}_j \in K_j(A, b)$
 \tilde{x}_{j+1} from \tilde{x}_j

Approx. theory

$$K_k = \text{span} \{ A^0 b, A^1 b, \dots, A^{k-1} b \}$$

$$x \in K_k \Rightarrow x = c_0 A^0 b + c_1 A^1 b + \dots + c_{k-1} A^{k-1} b \\ = p(A) b$$

$$\min_x \|Ax - b\| \\ \text{s.t. } x \in K_k(A, b)$$



$$\min_p \|p(A) b - b\| \\ \text{s.t. } p \in \mathcal{P}_{k-1}$$

$$A = V \Lambda V^{-1} \quad p(A) = V p(\Lambda) V^{-1}$$

$$\Rightarrow \min_{p \in \mathcal{P}_{k-1}}$$

$$\min \|q(\Lambda)\| \\ \text{s.t. } q \in \mathcal{P}_{k-1}, q(0) = -1$$

$$= \|V(p(\Lambda)V^{-1}b - V^{-1}b)\|$$

$$= \|V(p(\Lambda) - I)V^{-1}b\|$$

$$\leq \|V\| \|V^{-1}\| \|b\| \|p(\Lambda) - I\|$$

RHS

