Nov 6, 2020

$Ax = b$ $\quad x_0, x_1, \ldots, x_k, \ldots$ $\quad$ hope: $x_k \to A^{-1}b$

One key idea: only need "$y = Ax$" (not A_{ij})

Today: splitting/stationary methods

$A = M - N$ $\quad Ax = b$ $\quad (M-N)x = b \iff Mx_{k+1} = Nx_k + b$

$$x_{k+1} = M^{-1}N x_k + M^{-1}b = Rx_k + c$$

(special case of fixed point iteration: $x_{k+1} = F(x_k)$)

Convergence analysis

$$\begin{aligned} Mx_{k+1} &= Nx_k + b \\ - \quad Mx &= Nx + b \end{aligned}$$

$$M\underbrace{(x_{k+1} - x)}_{\hat{e}_{k+1}} = N\underbrace{(x_k - x)}_{\hat{e}_k}$$

$\hat{e}_{k+1} = R^{k+1} e_0$ $\quad \hat{e}_{k+1} = R\hat{e}_k$ $\quad (R = M^{-1}N)$

$$\begin{aligned} \|\hat{e}_{k+1}\| &\le \|R\| \, \|\hat{e}_k\| \\ &\le \|R\|^{k+1} \|\hat{e}_0\| \end{aligned}$$

Want: $\|R\| < 1$

Convergence

Any operator norm $\|\cdot\|$ s.t. $\|R\| < 1 \Rightarrow \hat{e}_k \to 0$ $x_k \to A^{-1}b$

$$\max_k |\lambda_k(R)| = \rho(R) \le \|R\|$$

$\exists \|\cdot\|_*$ s.t. $\|R\|_* \le \rho(R) + \varepsilon$

$\rho(R) < 1 \Rightarrow$ choose $\|\cdot\|_*$ so that $\|R\|_* < 1$ (convergence)

$\rho(R) \ge 1 \Rightarrow$ choose $\hat{e}_0 = x_0 - x$ with $R\hat{e}_0 = \lambda\hat{e}_0$ $|\lambda| = \rho(R)$

$$\|\hat{e}_{k+1}\| = \|R^{k+1}\hat{e}_0\| = \|\lambda^{k+1}\hat{e}_0\| = \rho(R)^{k+1}\|\hat{e}_0\| \ge \|\hat{e}_0\|$$

Gauss-Seidel!

$$\alpha(A_{kk} + x_k) = -\Sigma_{j \neq k} A_{kj} x_j + b_k$$