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$A = A^T, \quad A = Q \Lambda Q^T$

Power iteration

$\|x_0\|_2 = 1 \qquad |\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$

$y = A x_k$

$x_{k+1} = y / \|y\|_2 \qquad x_k = A^k x_0 / \|A^k x_0\|$

$x_0 = Q z \quad A^k x_0 = Q \Lambda^k z = \lambda_1^k Q \begin{pmatrix} z_1 \\ z_2 (\lambda_2/\lambda_1)^k \\ \vdots \\ z_n (\lambda_n/\lambda_1)^k \end{pmatrix}$

$A^k x_0 = \lambda_1^k \left[z_1 q_1 + z_2 (\lambda_2/\lambda_1)^k q_2 + \dots + (\lambda_n/\lambda_1)^k q_n \right]$

$\|A^k x_0\| = |\lambda_1|^k \sqrt{z_1^2 + z_2^2 (\lambda_2/\lambda_1)^{2k} + \dots + z_n (\lambda_n/\lambda_1)^{2k}}$

$\operatorname{sign}(\lambda_1) \cdot \operatorname{sign}(z_1) \cdot q_1$

$\|x_k - \pm q_1\|_2 = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right)$

Rayleigh Quotient Iteration

$\|x_0\|_2 = 1, \quad \lambda_0 = x_0^T A x_0$

$(A - \lambda_k I) y = x_k$

$x_{k+1} = y / \|y\|_2,$

$\lambda_{k+1} = x_{k+1}^T A x_{k+1}$

Suppose $Aq = \lambda q$

$\|x_k - q\| \leq \varepsilon$

$$\lambda_k = \frac{x_k^T A x_k}{x_k^T x_k} = R_A(x_k)$$

$$\nabla R_A(x) = \frac{1}{x^T x}\Big(\underbrace{(A+A^T)}_{=2A} x - 2R_A(x)x\Big)$$

$\nabla R_A(x^*) = 0 \iff Ax^* = R_A(x^*)\, x^*$

$$\underbrace{R_A(x_k)}_{\lambda_k} = R_A(q) + \cancel{\nabla R_A(q)^T}^{\,0} (x_k - q) + (x_k - q)^T H (x_k - q) \; O(\varepsilon^2) + \ldots \; O(\varepsilon^3)$$

$\Rightarrow |\lambda_k - \lambda| \lesssim O(\varepsilon^2)$

$\lambda_2[(A - \lambda_k I)^{-1}] / \lambda_1[(A - \lambda_k I)^{-1}]$

$$= \frac{q_n^T T_k q_n}{q_n^T q_n}$$

scalars