Oct 7, 2020

LLS: $\hat{x} = \arg\min_x \|Ax-b\|_2^2$ A $m \times n$ $m > n$
A full rank

$\hat{x} = A^+ b$ $\hat{x} = R^{-1} Q^T b$ $r = b - A\hat{x} = (I - QQ^T) b$

(1) conditioning: how sensitive problem is to perturbations

(2) stability: how accurate our algorithm is

backward stable alg $\Rightarrow$ relative error is $O(\kappa \cdot \varepsilon_{mach})$

LLS: $(A, b) \to (\hat{x}, A\hat{x}) = (A^+ b, AA^+ b)$

perturb A or b, how does $\hat{x}$ change? ($A\hat{x}$ change?)

$m = n \Rightarrow$ should get $\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = \sigma_1 / \sigma_n$

Ex: $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $b = \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix}$ $A^T A = [1]$ $A^T b = \varepsilon$

$\hat{x} = (A^T A)^{-1} A^T b = \varepsilon$

$b + \delta b = \begin{pmatrix} 2\varepsilon \\ 1 \end{pmatrix} \Rightarrow$ $\tilde{x} = 2\varepsilon$ $\|\tilde{x} - \hat{x}\| / \|\hat{x}\| = 1$

$\|\delta b\| / \|b\| = O(\varepsilon)$

Problem: Projection of b onto range(A) is small $(O(\varepsilon))$
relative to the size of b

$A\hat{x} = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}$ $\quad A\tilde{x} = \begin{pmatrix} 2\varepsilon \\ 0 \end{pmatrix}$ $\quad \dfrac{\| A\tilde{x} - A\hat{x} \|}{\| A\hat{x} \|} = O(1)$

$\hat{x} = R^{-1} Q^T b$

$k_2(A) = \sigma_1 / \sigma_n$ $\qquad \boxed{A} = \boxed{U} \underbrace{\boxed{\diagdown}}_{\Sigma} \boxed{V^T}$

problems

$\|A\|_2 =$

small perturbations to A can cause relatively large changes to

$r = A\hat{x} - b$

$\text{range}(A) \iff A^T b = 0 \iff U^T b = 0$

not much to do