

Oct 2, 2020

Last time: linear LS: $\hat{x} = \arg \min_x \|Ax - b\|_2^2$

$$\hat{x} = A^+ b = (A^T A)^{-1} A^T b = V \Sigma^{-1} U^T b = R^{-1} Q^T b$$

$$\begin{matrix} \vec{a} \\ \boxed{A} \end{matrix} \equiv \begin{matrix} \vec{a} \\ \boxed{Q} \end{matrix} \begin{matrix} \vec{r} \\ \boxed{R} \end{matrix} \quad Q^T Q = I$$

$$\begin{pmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_n \end{pmatrix} = \begin{pmatrix} \vec{q}_1 \\ \vdots \\ \vec{q}_n \end{pmatrix} \begin{pmatrix} r_{11} & \dots & r_{1n} \\ & r_{22} & \dots \\ & & \ddots \\ & & & r_{nn} \end{pmatrix}$$

$$\begin{aligned} a_1 &= q_1 r_{11} & q_1 &= \frac{a_1}{\|a_1\|_2} & q_1^T q_1 &= \frac{a_1^T a_1}{\|a_1\|_2^2} = 1 \\ a_2 &= q_1 r_{12} + q_2 r_{22} & r_{11} &= \|a_1\|_2 \end{aligned}$$

$$q_1^T a_j = q_1^T q_1 r_{1j} + \dots + q_2^T q_1 r_{2j}$$

$$q_1^T a_2 = \cancel{q_1^T} q_1 r_{12} + \cancel{q_2^T} q_1 r_{22} \quad r_{12} = q_1^T a_2$$

$$\underbrace{a_2 - q_1 r_{12}}_{v_2} = q_2 r_{22} \quad q_2 = \frac{v_2}{\|v_2\|_2} \quad r_{22} = \|v_2\|_2$$

$$r_{ij} = q_i^T a_j$$

(Classical) Gram-Schmidt

for $j=1:n$

$$v_j = a_j$$

for $i=1:j-1$

$$r_{ij} = q_i^T a_j \quad \text{O}(m)$$

$$v_j = v_j - r_{ij} q_i$$

$$r_{jj} = \|v_j\|_2$$

$$q_j = v_j / r_{jj}$$

$$r_{jj} = 0?$$

r_{jj} close to 0?

$O(n)$ iterations

$O(m \cdot n^2)$ total

$$[a_1 \ A_2] = [q_1 \ Q_2] \begin{bmatrix} r_{11} & r_{12}^T \\ 0 & R_{22} \end{bmatrix}$$

$$a_1 = q_1 r_{11} \quad q_1 = \frac{a_1}{\|a_1\|_2} \quad r_{11} = \|a_1\|_2$$

$$q_1^T A_2 = \cancel{q_1^T} q_1 r_{12}^T + \cancel{q_1^T} Q_2 R_{22}$$

$$r_{12}^T = q_1^T A_2$$

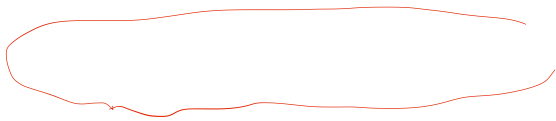
$$A_2 - q_1 r_{12}^T = Q_2 R_{22}$$

Modified Gram-Schmidt

for $j=1:n$

$$r_{jj} = \|A(:,j)\|_2$$

$$(I - P)^2 = I - 2P + P^2$$



P^2

