Oct 2, 2020

Last time: linear LS: $\hat{x} = \arg\min_x \|Ax-b\|_2^2$

$$\hat{x} = A^+ b = (A^TA)^{-1}A^T b = V\Sigma^{-1}U^T b = R^{-1}Q^T b$$

$$\underset{m\times n}{A} = \underset{m\times n}{Q}\ \underset{n\times n}{R} \qquad Q^TQ = I$$

$$\begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} r_{11} & \cdots & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ & 0 & \ddots & \vdots \\ & & & r_{nn} \end{pmatrix}$$

$$a_1 = q_1 r_{11} \longrightarrow q_1 = \frac{a_1}{\|a\|_2} \qquad q_1^T q_1 = \frac{a_1^T a_1}{\|a\|_2^2} = 1$$

$$r_{11} = \|a\|_2$$

$$a_2 = q_1 r_{12} + q_2 r_{22}$$

$$\vdots$$

$$q_1^T a_2 = \overbrace{q_1^T q_1}^{1} r_{12} + \overbrace{\cancel{q_1^T q_2}}^{0} r_{22} \qquad r_{12} = q_1^T a_2$$

$$\underbrace{a_2 - q_1 r_{12}}_{v_2} = q_2 r_{22} \qquad q_2 = \frac{v_2}{\|v_2\|_2}$$

$$r_{22} = \|v_2\|_2$$

$$q_i^T a_j = \overbrace{q_i^T q_i}^{1} r_{ij} + \cdots + \overbrace{\cancel{q_i^T q_j}}^{0} r_{jj}$$

$$\vdots$$

$$r_{ij} = q_i^T a_j$$

(Classical) Gram-Schmidt

```
for j = 1:n
    v_j = a_j
    for i = 1:j-1
        r_ij = q_i^T a_j      | O(m)
        v_j -= r_ij q_i
    r_jj = ||v_j||_2
    q_j = v_j / r_jj
```

$r_{jj} = 0$?

r_{jj} close to 0?

$O(n)$ iterations

$O(m \cdot n^2)$ total

$$\begin{bmatrix} a_1 & A_2 \end{bmatrix} = \begin{bmatrix} q_1 & Q_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12}^T \\ 0 & R_{22} \end{bmatrix}$$

$$a_1 = q_1 r_{11} \qquad q_1 = \frac{a_1}{\|a_1\|_2} \qquad r_{11} = \|a_1\|_2$$

$$q_1^T A_2 = \overbrace{\cancel{q_1^T q_1}}^{1} r_{12}^T + \overbrace{\cancel{q_1^T Q_2}}^{0} R_{22}$$

$$r_{12}^T = q_1^T A_2$$

$$A_2 - q_1 r_{12}^T = Q_2 R_{22}$$

Modified Gram-Schmidt

```
for j = 1:n
    r_jj = ||A(:,j)||_2
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$$(I-P)^2 = I - 2P + P^2$$

Pv

b